Ph 12c

Homework Assignment No. 1
Due: 8pm Thursday, 7 April 2016

Do Problem 3 in Chapter 2 of Kittel and Kroemer, plus these three additional problems:

1. The moment-generating function and the central limit theorem. Suppose that x is a random variable taking values on the real line, and $p(x)$ is a probability distribution for x. We say that

$$X_n \equiv \langle x^n \rangle = \int_{-\infty}^{\infty} dx\ p(x) x^n$$

is the nth *moment* of the probability distribution, and that

$$\bar{X}(t) = \langle e^{tx} \rangle = \sum_{n=0}^{\infty} \frac{X_n t^n}{n!}$$

is the *moment-generating function* of the distribution.

a) Compute the moment-generating function for the normalized Gaussian distribution with mean zero and variance σ^2,

$$q(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}. \tag{1}$$

(Note that it is easy to do the integral $\bar{X}(t) = \int_{-\infty}^{\infty} dx\ q(x) e^{tx}$ by shifting the integration variable by a constant.)

b) By expanding $\bar{X}(t)$ in a power series, show that for the normalized Gaussian distribution $\langle x^n \rangle = 0$ for n odd, and find an expression for $\langle x^{2n} \rangle$ for each nonnegative integer n.

c) Now suppose that $\{x_1, x_2, x_3, \ldots, x_N\}$ are independent random variables, all identically distributed with probability distribution $p(x)$. Consider the random variable

$$u_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i,$$

which (aside from the non-standard but conveniently chosen normalization), represents the result of sampling the same distribution N times and averaging the results. The moment generating function for u_N is

$$\bar{U}_N(t) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_N \; p(x_1)p(x_2)\cdots p(x_N) \; e^{tu_N};$$

express $\bar{U}_N(t)$ in terms of $\bar{X}$, the moment generating function for the distribution $p(x)$.

d) Assuming that the distribution $p(x)$ has mean zero ($\langle x \rangle = 0$), show that $\bar{U}_N(t)$ can be approximated as

$$\bar{U}_N(t) \approx \left(1 + \frac{t^2}{2N} X_2 + O\left(N^{-3/2}\right)\right)^N ,$$

and show that in the limit $N \to \infty$, $\bar{U}_N(t)$ becomes the moment-generating function of a Gaussian distribution with mean zero. What is the variance of this Gaussian?

2. Biased coin. When a biased coin is flipped the outcome is heads with probability p and tails with probability $1-p$. If this coin is flipped N times, the probability that the total number of heads is n is

$$p(n) = \binom{N}{n} p^n (1-p)^{N-n}.$$

The most likely value of n is $n = pN$, but there are fluctuations about this most likely value.

Denote $n = Np + s$, and suppose that $N \gg 1$. In this limit, $p(n)$, regarded as a function of s, approaches a Gaussian with mean zero and some variance σ_p^2; hence,

$$\ln[p(n)] = \text{constant} - \frac{s^2}{2\sigma_p^2} + O\left(s^4\right),$$

where "constant" means a term independent of s. Calculate σ_p^2 using the Stirling approximation and the approximations $s \ll pN$ and $s \ll (1-p)N$. To save work, notice that you only need to find the coefficient of s^2 in the expansion of $\ln[p(n)]$; you don't need to worry about the constant terms or the linear terms. Compare your value of σ_p^2 with the result $\sigma^2 = N/4$ found in class for the case $p = 1/2$.

3. Probability of a large deviation. For the Gaussian distribution Eq. (1), x is not likely to deviate from zero by an amount much larger than σ. To estimate the probability of a large deviation, we observe that the probability for x to have a value exceeding t,

$$P(x \geq t) = \int_t^\infty dx \; q(x),$$

has an asymptotic expansion for $t^2 \gg \sigma^2$.

a) Integrate by parts repeatedly to show that

$$P(x \geq t) = \sqrt{\frac{\sigma^2}{2\pi t^2}} \; e^{-t^2/2\sigma^2} \left(A - B\frac{\sigma^2}{t^2} + O\left(\frac{\sigma^4}{t^4}\right)\right),$$

where A and B are positive constants, and find A and B.

b) Estimate the probability that x is 10σ or larger.