Ph 12c

Homework Assignment No. 4
Due: 8pm, Thursday, 28 April 2016

Do Problem 12 in Chapter 5 plus these additional problems:

1. Isentropic model of the atmosphere.

a) Consider an ideal gas of particles in the earth's gravitational field, where each gas molecule has mass m and g is the acceleration due to gravity. The dependence of the pressure $p(z)$ on the height z is determined by the condition for mechanical equilibrium: for the gas contained in a small region, the downward gravitational force is compensated by the difference between the pressure at the bottom of the region and the pressure at the top. Use this condition to express dp/dz in terms of m, g, p, and the temperature τ.

b) Suppose that the entropy per particle in the earth's atmosphere is independent of altitude, so that $pn^{-\gamma}$ is a constant, where $n = N/V$ is the concentration, the number of gas molecules per unit volume. Use the ideal gas law $\tau = p/n$ and the result of (*a*) to show that $d\tau/dz$ is a constant that can be expressed in terms of γ, m, and g.

c) Assuming $\gamma = 7/5$, find dT/dz in °C per km, where m is the mass of a nitrogen molecule.

2. Fluctuations in particle number.

Using the Gibbs distribution, the expectation value for the particle number N of a system in thermal and diffusive contact with a reservoir at temperature τ and chemical potential μ is

$$\langle N \rangle = \mathcal{Z}^{-1} \sum_s N_s \exp\left((N_s\mu - \epsilon_s)/\tau\right),$$
$$\mathcal{Z}(\tau, \mu) = \sum_s \exp\left((N_s\mu - \epsilon_s)/\tau\right).$$

a) Show that

$$\langle (\Delta N)^2 \rangle \equiv \langle (N - \langle N \rangle)^2 \rangle = \tau \left(\frac{\partial \langle N \rangle}{\partial \mu}\right)_\tau .$$

b) For a classical ideal gas, show that

$$\frac{\langle (\Delta N)^2 \rangle}{\langle N \rangle^2} = \frac{1}{\langle N \rangle}.$$

Hence, for $N \gg 1$, the fluctuations in particle number are small compared to the mean number of particles.

3. Ideal gas adsorption

Using the distribution function $f(\epsilon) = e^{(\mu-\epsilon)/\tau}$, the particle number N of a classical ideal gas in a box with volume V can be expressed as

$$N = e^{\mu/\tau} \sum_a e^{-\epsilon_a/\tau} = e^{\mu/\tau} \left(n_Q V \right),$$

where the sum is over all single-particle orbitals, and $n_Q = \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2}$ is the quantum concentration. Hence the chemical potential is

$$\mu = \tau \ln \left(n/n_Q \right),$$

where n is the concentration of particles per unit volume.

a) Use the same method to find the chemical potential of a two-dimensional classical ideal gas, expressed in terms of the concentration $\bar{n}$ of particles per unit *area*.

Suppose that the walls of a box containing a classical ideal gas with temperature τ can adsorb the gas particles, where it costs energy Δ to remove an adsorbed particle from the wall. The adsorbed particles move freely along the walls, and so can be modeled as an ideal gas confined to two dimensions. The adsorbed particles are in thermal and diffusive equilibrium with the gas particles contained inside the box.

b) The concentration $\bar{n}$ of adsorbed particles per unit area is related to the concentration n of gas particles per unit volume according to

$$\bar{n} = Cn$$

where C depends on Δ, τ, and the particle mass m. Find C.

4. Diffusive contact

Two identical systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are both in thermal contact with a large reservoir and in diffusive contact with one another. For both systems, the free energy F is related to the particle number N by $F = cN^2$, where c is an N-independent constant (the same constant for both systems).

(*a*) (20 points) A battery maintains a chemical potential difference

$$\Delta = \mu_{2,\mathrm{ext}} - \mu_{1,\mathrm{ext}} > 0$$

between the two systems. In diffusive equilibrium, find the number N_1 of particles in $\mathcal{S}_1$ and the number N_2 of particles in $\mathcal{S}_2$, expressed in terms of Δ, c, and the total particle number $N = N_1 + N_2$.

(*b*) (20 points) Now the battery is disconnected, and useful work is extracted isothermally as the particles flow slowly from $\mathcal{S}_1$ to $\mathcal{S}_2$ until diffusive equilibrium is reestablished. How much work is extracted?