Ph 12c

Homework Assignment No. 5
Due: 8pm, Thursday, 12 May 2016

1. Light bulb in a refrigerator

A refrigerator that draws 50 W of power is contained in a room at temperature 300°K. A 100 W light bulb is left burning inside the refrigerator. Find the steady-state temperature inside the refrigerator assuming it operates reversibly and is perfectly insulated.

2. Imperfectly insulated refrigerator

An ideal (reversible) air conditioner cools a room to temperature $\tau_m < \tau_h$, where τ_h is the temperature outdoors. Meanwhile, an ideal (reversible) refrigerator in the room cools drinks to temperature $\tau_l < \tau_m$. Because the refrigerator is not perfectly insulated, heat flows into the refrigerator from the room at the rate $\dot{Q}_1 = A_1(\tau_m - \tau_l)$, and because the room is not perfectly insulated, heat flows into the room from outdoors at the rate $\dot{Q}_2 = A_2(\tau_h - \tau_m)$. In the steady state, heat is removed from the refrigerator at the same rate it flows in. Heat is removed from the room by the air conditioner and also by the flow of heat from the room into the refrigerator; in the steady state the total rate of heat removal from the room matches the total rate for heat entering the room due to exhaust from the refrigerator combined with the flow of heat from outdoors.

(*a*) In the steady state, find the total power P_{total} (rate of work done per unit time) expended by the refrigerator and air conditioner together. As a check, verify that your answer makes sense for $\tau_m = \tau_h$ and for $\tau_m = \tau_l$.

(*b*) Find the optimal value τ_m^{opt} of τ_m which minimizes P_{total}. As a check, verify that $\tau_m^{\text{opt}} = \tau_l$ in the case $\tau_h = \tau_l$. Also check that your answer makes sense in the limit $A_1 \gg A_2$ and in the limit $A_1 \ll A_2$. Explain.

(*c*) Assuming $A_1 = A_2$, find the optimal value of P_{total}. Check your answer by verifying that $P_{\text{total}} = 0$ for $\tau_h = \tau_l$.

3. Photonic heat engine

Consider a heat engine undergoing a Carnot cycle, where the working fluid is a photon gas rather than a classical ideal gas. In the first stroke the gas expands isothermally at temperature τ_h from the initial volume V_1 to the final volume V_2. In the second stroke it expands isentropically to volume V_3, cooling to temperature τ_l. In the third stroke it is compressed isothermaly at temperature τ_l to volume V_4, and in the fourth stroke it is compressed isentropically back to volume V_1, heating to temperature τ_h.

(a) The energy per unit volume of a photon gas is $U/V = A\tau^4$, where $A = \pi^2/15\hbar^3c^3$. Use the thermodynamic identity

$$dU = \tau d\sigma - PdV$$

to find the entropy σ of the gas, expressed in terms of A, τ, and V. Assume that the entropy is zero at $\tau = 0$.

(b) Use the thermodynamic identity again to express the pressure P in terms of A, τ, and V.

(c) Calculate the work done W_{12} and the heat added Q_{12} during the first stroke of the cycle, expressed in terms of A, τ_h, V_1 and V_2. Verify that $Q_{12} - W_{12}$ is the change in the internal energy of the gas.

(d) Express the work W_{34} done by the gas in the third stroke (a negative number), in terms of A, τ_l, V_3 and V_4.

(e) Use the condition $\sigma =$ constant during the isentropic strokes to express V_3 and V_4 in terms of τ_h, τ_l, V_1, and V_2.

(f) Find the work W_{23} done during the second stroke and the work W_{41} done during the fourth stroke.

(g) Express the net work $W = W_{12} + W_{23} + W_{34} + W_{41}$ done during the complete cycle in terms of A, τ_h, τ_l, V_1 and V_2. Comparing to Q_{12}, check that the engine achieves the ideal Carnot efficiency.

4. Bose condensation in two dimensions

Consider an ideal gas of non-relativistic spin-0 bosons, at temperature τ, in a *two–dimensional* box of side L.

(a) Find the *two–dimensional* density of states factor $\mathcal{D}(\varepsilon)$.

(b) Express the activity $\lambda \equiv e^{\mu/\tau}$ in terms of N_0, the number of particles in the ground orbital. Use the convention that the energy of the ground orbital is $\epsilon_0 = 0$.

(c) Find $N_e(\tau)$, the number of particles in excited orbitals. You may assume that the box is big enough so that the sum over states can be replaced by an integral. Be sure to use the formula found in (b) for λ, not the $N_0 \to \infty$ limit of that formula. Your answer for N_e will therefore be expressed in terms of N_0. **Hint**: $\int dx (ae^x - 1)^{-1} = \ln(a - e^{-x})$.

(d) Find the *two–dimensional* Einstein condensation temperature τ_E. This is the smallest temperature such that, for $\tau > \tau_E$, the fraction $N_0/(N_0 + N_e)$ of particles in the ground orbital vanishes in the limit $L \to \infty$. (The limit is to be taken with the density $(N_0 + N_e)/L^2$ held fixed.)

5. Heat capacity of graphene

Geim and Novoselov received the 2010 Nobel Prize in Physics for their studies of graphene, a single layer of carbon atoms bonded into a two-dimensional hexagonal lattice. Remarkably, electrons in graphene behave like relativistic massless fermions; for each value of the wavenumber $\vec{k} = (k_x, k_y)$, there are two single-particle orbitals, with energies

$$\epsilon_\pm(\vec{k}) = \pm\hbar v|\vec{k}|.$$

The Fermi energy is $\epsilon_F = 0$; hence at zero temperature the orbitals with negative energy are occupied, and the orbitals with positive energy are empty.

Assuming the electrons can be treated as an ideal gas, and that there are two spin states for each orbital, the internal energy of the electrons has the form

$$U(\tau) - U(0) = \frac{1}{3}\gamma A \tau^3,$$

where A denotes the area, and hence the electron heat capacity is $C = \gamma A \tau^2$. Find γ. (**Hint**: $\int_0^\infty dx\ x^2/(e^x + 1) = 1.803$.)