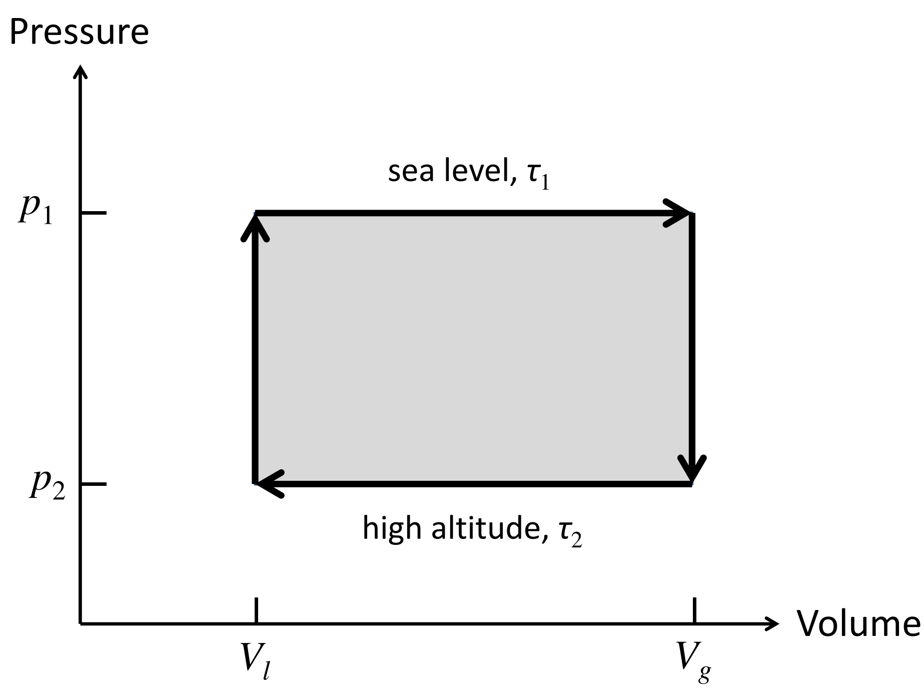
Ph 12c

Homework Assignment No. 7
Due: Thursday, 26 May 2016, 8 pm

1. Hurricane!

A hurricane is a heat engine, powered by the warm surface of the ocean, which undergoes a four-stroke cycle sketched in the figure. (1) Water evaporates from the ocean surface at temperature τ_1 and pressure p_1 (2) Water vapor rises to high altitude, where the temperature is $\tau_2 < \tau_1$ and the pressure is $p_2 < p_1$. (3) Water vapor condenses at temperature τ_2 and pressure p_2. (4) Rain falls from high altitude to the ocean surface.



Suppose that a mole of liquid water occupies volume V_l both at sea level and at high altitude and that a mole of water vapor occupies volume V_g both at sea level and at high altitude. (We assume that any change in volume as the vapor rises or the water falls is negligible.) Suppose that the latent heat per mole required to convert liquid water to water vapor at sea level is L.

a) Express the work done on a mole of water during one cycle in terms of V_l, V_g, p_1, p_2, τ_1, and τ_2.

b) Assuming that this heat engine achieves the Carnot efficiency, find an expression for $(p_1 - p_2)/(\tau_1 - \tau_2)$. Derive the Clausius-Clapeyron relation by considering the limit $(\tau_1 - \tau_2) \to 0$.

c) If the hurricane has Carnot efficiency, generates power W in Watts, and has area A in m^2, at what rate is water evaporating from the ocean surface? Give your answer as the height h in meters of the layer of water that evaporates in one day, expressed in terms of W, A, L, τ_1, and τ_2. (A similar amount of water falls as rain.)

d) A powerful hurricane generates about 200 terawatts (2×10^{14} W) and covers an area of about 10^5 km^2. (In contrast, the world's power demand is currently only about 18 terawatts.) The ocean surface has temperature $T_1 = 27°$C and the upper atmosphere has temperature $T_2 = -73°$C. The latent heat of vaporization of water is 41 kJ per mole. What is h?

2. Scaling hypothesis from Landau theory

In the Landau theory of second-order phase transitions, the Helmholtz free energy close to the critical point can be expressed as a function of the temperature τ and order parameter ξ in the form

$$F(\epsilon, \xi) = \frac{1}{2}A\epsilon\xi^2 + \frac{1}{4}B\xi^4, \tag{1}$$

where A and B are positive constants and

$$\epsilon = \frac{\tau - \tau_c}{\tau_c} \tag{2}$$

denotes the dimensionless deviation of the temperature from the critical temperature τ_c. If λ is an applied external field that couples to the order parameter (for example, the applied magnetic field H), then we can perform a *Legendre transform* to obtain the Gibbs free energy $G(\epsilon, \lambda)$, a function of the temperature and external field:

$$G(\epsilon, \lambda) = \left[F(\epsilon, \xi) - \lambda\xi\right]_{\text{stat wrt }\xi} . \tag{3}$$

That is, the Gibbs free energy is obtained by evaluating the expression in square brackets at the value of ξ where this expression is stationary with respect to ξ.

We say that the Gibbs free energy $G(\epsilon, \lambda)$ satisfies the *scaling hypothesis* if, when ϵ and λ are small,

$$G(\Omega^p \epsilon, \Omega^q \lambda) = \Omega \; G(\epsilon, \lambda) \; ; \tag{4}$$

here Ω is a real "scaling variable," and p, q are (not necessarily integer) exponents that encode the scaling properties of the system near the critical point. Show that the scaling hypothesis is satisfied in Landau theory for particular values of p and q, and find these values.

(To perform the Legendre transform explicitly, you would need to solve a cubic equation. However, it is not necessary to perform the Legendre transform explicitly to solve this problem. It is useful to notice that the formula for $G(\epsilon, \lambda)$ in eq.(3) would be unchanged if we rescaled ξ by a constant inside the square brackets, since we eliminate ξ anyway when we evaluate the quantity in square brackets at its stationary point.)

3. Critical exponents from the scaling hypothesis

In many cases of interest, Landau theory fails, but the scaling hypothesis still holds with different values of p and q than the values predicted by Landau theory. Then various other critical exponents can be expressed in terms of p and q.

For example, consider the isothermal susceptibility

$$\chi_\tau = \left(\frac{\partial \xi}{\partial \lambda} \right)_\tau \; , \tag{5}$$

where the order parameter ξ is

$$\xi = - \left(\frac{\partial G}{\partial \lambda} \right)_\tau \; . \tag{6}$$

The critical exponent γ is defined by the behavior of χ_τ for $\lambda = 0$ and ϵ close to zero:

$$\chi_\tau \sim |\epsilon|^{-\gamma} \; . \tag{7}$$

From $\chi_\tau = - \left(\partial^2 G / \partial \lambda^2 \right)_\tau$ and the scaling hypothesis, we find

$$\Omega^{2q} \chi_\tau(\Omega^p \epsilon) = \Omega \; \chi_\tau(\epsilon) \; . \tag{8}$$

Now we may choose Ω so that as ϵ approaches zero, $\Omega^p\epsilon$ is held fixed (hence $\Omega \propto \epsilon^{-1/p}$); then

$$\chi_\tau \sim \Omega^{2q-1} \sim \epsilon^{-(2q-1)/p} \ . \tag{9}$$

We conclude that

$$\gamma = \frac{2q-1}{p} \ . \tag{10}$$

Other critical exponents can be derived by similar reasoning.

a) The exponent β describes how the order parameter ξ turns on for τ slightly less than τ_c and $\lambda = 0$:

$$\xi \sim (-\epsilon)^\beta \ . \tag{11}$$

Express β in terms of p and q.

b) The exponent δ describes the relation between the order parameter ξ and the external field λ on the critical isotherm $\epsilon = 0$:

$$\lambda \sim \xi^\delta \ . \tag{12}$$

Express δ in terms of p and q.

c) The exponent α describes how the heat capacity $C_\lambda = \tau(\partial\sigma/\partial\tau)_\lambda$ diverges as ϵ approaches zero for $\lambda = 0$:

$$C_\lambda \sim |\epsilon|^{-\alpha} \ . \tag{13}$$

Express α in terms of p and q.

d) Check your answers for parts (a),(b),(c), and Problem 2 by verifying the Landau theory exponents: $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$.

Because we have computed $\alpha, \beta, \gamma, \delta$ in terms of p and q, we obtain two relations among the exponents.

e) Express α in terms of β and δ. (This is called the *Griffiths relation.*)

f) Express γ in terms of α and β. (This is called the *Rushbrooke relation.*)

4. Equation of state from the scaling hypothesis

a) The equation of state is a relation among the variables ξ, λ and τ, where ξ is the order parameter

$$\xi = -\left(\frac{\partial G}{\partial \lambda}\right)_\tau . \tag{14}$$

The purpose of this problem is to show that the scaling hypothesis requires the equation of state to have a special form in the vicinity of the critical point $\epsilon = \lambda = 0$.

Using the scaling hypothesis and assuming $\epsilon > 0$, show ξ as a function of ϵ and λ satisfies the relation

$$\frac{\xi(\epsilon, \lambda)}{\epsilon^a} = f\left(\lambda/\epsilon^b\right) , \tag{15}$$

for some function f, and express the exponents a and b in terms of β and δ. (You are not being asked to determine the function f.)

Thus, if $\tilde{\xi} \equiv \xi/\epsilon^a$ is plotted as a function of $\tilde{\lambda} \equiv \lambda/\epsilon^b$, data for different positive values of ϵ lie on the same curve. This prediction is well confirmed in various experimental studies of second-order phase transitions.

b) When an external field λ is applied, the field and order parameter are related by

$$\lambda = \frac{\partial}{\partial \xi} F(\epsilon, \xi). \tag{16}$$

Using the Landau theory expression for $F(\epsilon, \xi)$ from Problem 2, show that the order parameter λ as a function of ϵ and ξ satisfies the relation

$$\frac{\lambda(\epsilon, \xi)}{\epsilon^b} = h(\xi/\epsilon^a); \tag{17}$$

find the values of a and b, and find the function h in terms of the parameters A and B in $F(\epsilon, \xi)$. As a check, compare your result with the values of a and b found in part (*a*), using the Landau theory exponents q and p found in Problem 2.