Ferromagnetism

Recall, a magnetic field applied to a magnetic moment tends to align the moment with the field (exerts torque)

$\vec{B} \uparrow$ $\quad \vec{\mu}$ $\qquad E = -\vec{B}\cdot\vec{\mu}$ $\qquad$ (Minimize energy by lining up)

current loop-or spin

So current loops reinforce the applied field

"Magnetization" $\quad \vec{M}$ = magnetic moment / volume

$$\vec{M} = \chi_m \vec{H} \quad \text{applied field}$$

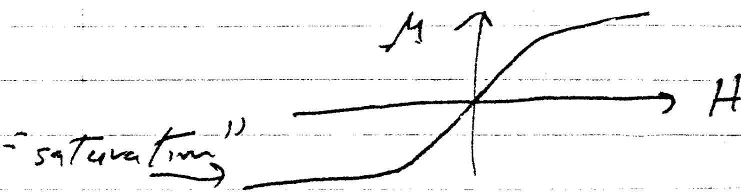
χ_m = "magnetic susceptibility"

so — total magnetic field given by $\vec{B} = \vec{H} + 4\pi\vec{M}$

(really average of a field that is inhomogeneous) $\rightarrow$ $\vec{B} = \vec{H} + 4\pi\chi_m\vec{H} = \mu\vec{H}$

μ = magnetic permeability

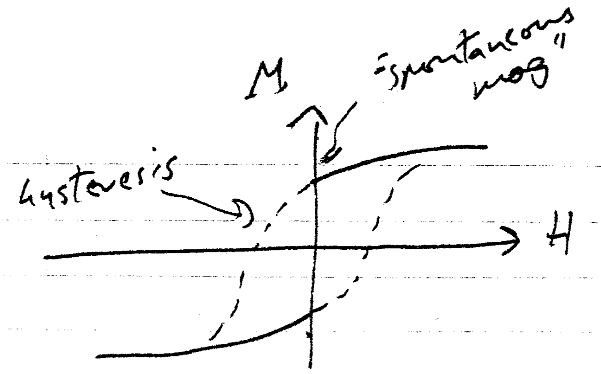
Linear relation actually applies only for weak applied field



Paramagnetism — magnetization aligns with H, but turns off when $H \to 0$

In some magnetic materials (e.g. iron), at sufficiently low temperature, M remains nonzero as $H \to 0$ ($T_c = 770°C$ for Fe; curie point)

(Ferromagnetism — spontaneous magnetization)



So — M is discontinuous as a function of H

— example of 1st order phase transition

(Analogous to jump in volume at given pressure, in gas-liquid transition)

In practice — metastability. Real magnets exhibit hysteresis

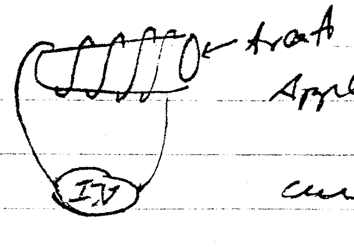
Claim: H analogous to pressure

M ($\times$ Volume) analogous to volume

$$\chi_T = \left(\frac{\partial M}{\partial H}\right)_T$$ — analogous to compressibility

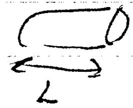
Strengthen this analogy by considering "magnetic work"

— $p\,dV \sim (\text{Vol}) \times H\,dM$



Apply H by putting material in a solenoid

$$\text{curl}\, H = \frac{4\pi}{c} J$$



or $HL = \frac{4\pi}{c} n I$ (n = no. of turns)

$$H = \frac{4\pi}{c} n I / L$$

voltage due to back emf

$$\text{curl}\, E = -\frac{1}{c}\frac{\partial B}{\partial t} \Rightarrow V = \frac{1}{c}\frac{d}{dt}(\text{flux})$$

$$= \frac{1}{c}\, n\,(\text{Area})\,\frac{\partial B}{\partial t} \quad \text{(Faraday)}$$

Power $$\frac{dW}{dt} = IV = \frac{c}{4\pi}\frac{L}{n} H \frac{1}{c} n A \frac{\partial B}{\partial t} = \frac{\text{Vol}}{4\pi} H \frac{\partial B}{\partial t}$$

10.25

So -- $dW = \frac{V}{4\pi} H\,dB = \frac{V}{4\pi} H\,d(H + 4\pi M)$

$$= V\Big(\underbrace{\tfrac{1}{8\pi}\,dH^2}_{\text{stored in H field}} + \underbrace{H\,dM}_{\text{work done on sample}}\Big)$$

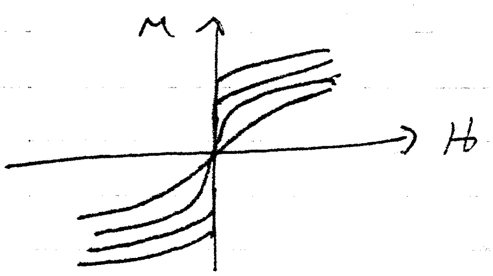
$$\boxed{dW = V(H\,dM)}$$

H, like p, is applied

M, like V, is response

χ, like K, tells how "stiff" is response

Heat up -- How do isotherms behave?



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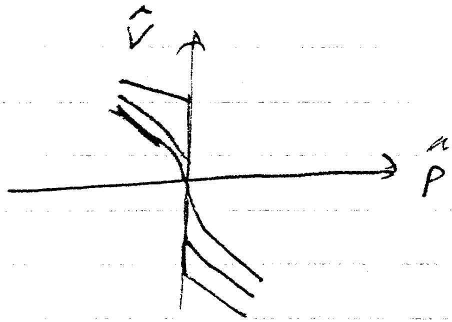
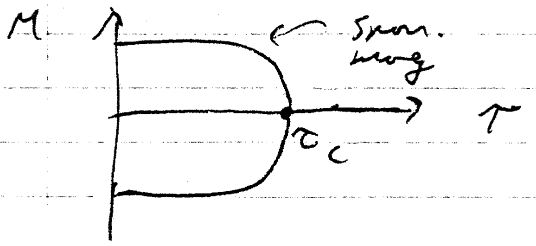
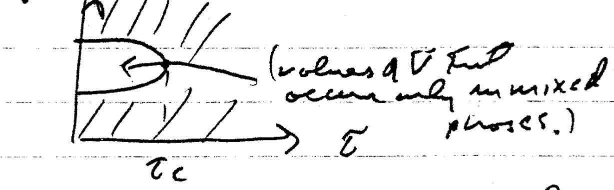
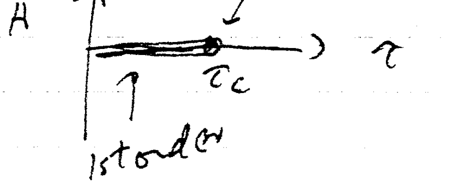


Behavior at zero H







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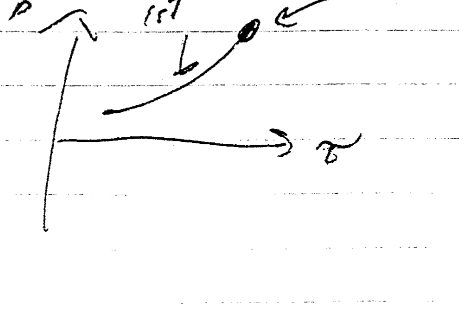


We want to develop analog of Van der Waals (1873) theory (Pierre Weiss – 1907)

First — simple model of paramagnetism (alignment of spins by applied field)
spin has magnetic moment μ

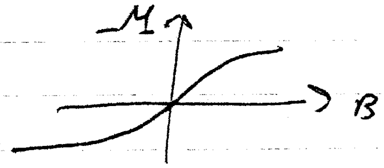
$\Delta E = 2\mu B$

Boltzmann $\Rightarrow$ $\dfrac{n_\uparrow}{n_\downarrow} = e^{-2\mu B/\tau}$

Fractional excess $\dfrac{n_\downarrow - n_\uparrow}{n_\uparrow + n_\downarrow} = \dfrac{1 - e^{-2\mu B/\tau}}{1 + e^{-2\mu B/\tau}} = \tanh(\mu B/\tau)$

So.. if n = no/volume of spins, we have

$$M = n\mu \tanh(\mu B/\tau)$$



Here B is the magnetic field seen by a spin — which depends on magnetization as well as applied field

Model: $B_{eff} = H + \lambda M$
("mean field theory")

($\lambda \neq 4\pi$, because this is not average B, but B at location of spin)

So — relation between H and M becomes nonlinear

$$M = n\mu \tanh\left[\frac{\mu}{\tau}(H + \lambda M)\right]$$

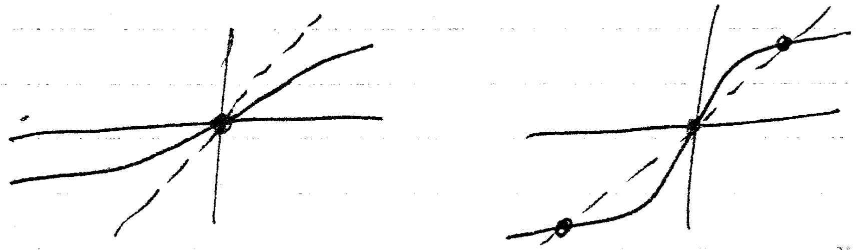
How do isotherms behave?

Look at $H = 0$ (turn off applied field)

Let $m = M/n\mu$

$$m = \tanh\left(\frac{n\mu^2}{\tau}\lambda m\right)$$

either one or three solutions, depending on slope at the origin

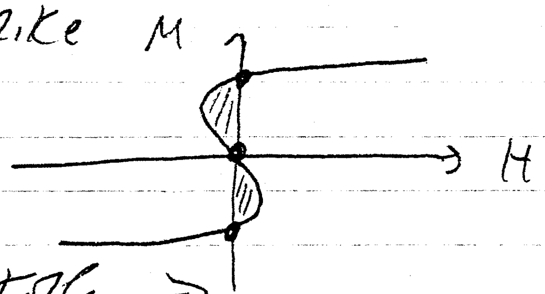


solutions with "spontaneous magnetization" turn on for

$$\frac{\lambda n\mu^2}{\tau} > 1$$

or $\boxed{\tau < \tau_c = \lambda n\mu^2}$

For $\tau < \tau_c$, isotherm looks like



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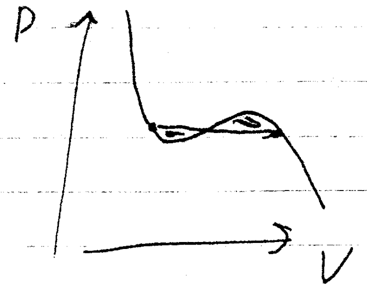
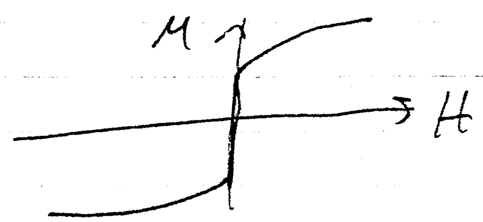
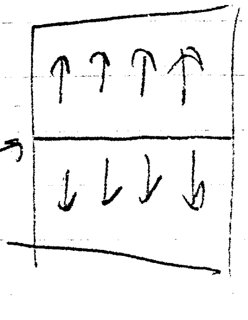


Unstable →

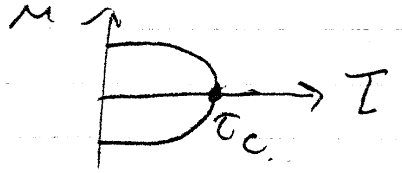
Maxwell construction tells us there is 1st order phase transition at $H = 0$.



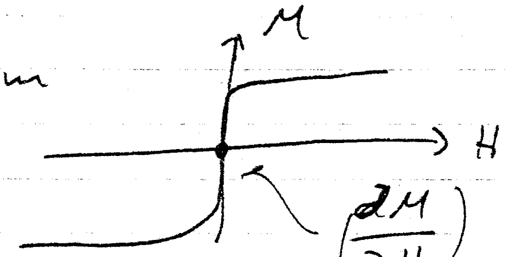


coexistent phases

We have spontaneous magnetization $(H=0)$

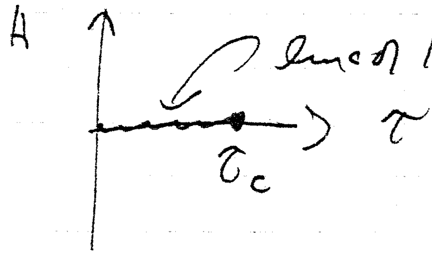


critical isotherm



$$\left(\frac{\partial M}{\partial H}\right)_\tau = \chi_m \to \infty$$

divergent susceptibility



line of 1st order phase transitions ends

Critical Exponents

How does magnetization turn on?

We have $m = \tanh\left(\frac{n\mu^2}{\tau}\lambda m + \frac{\mu}{\tau}H\right)$

Rescale temperature $\hat{\tau} = \tau/\tau_c = \tau/\lambda n\mu^2$

consider $H=0$

$$\Rightarrow \quad m = \tanh\left(\frac{m}{\hat{\tau}}\right)$$

For $\hat{\tau}$ near 1, soln is at $m \ll 1$, and we can expand

$$\tanh x \simeq x - \tfrac{1}{3}x^3 + \cdots$$

$$\Rightarrow m = m/\hat{\tau} - \tfrac{1}{3}\, m^3/\hat{\tau}$$

Write $\hat{\tau} = 1 + \delta\hat{\tau}$

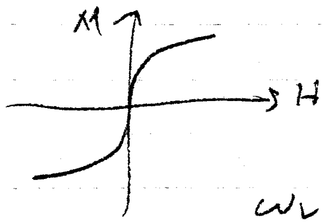
$$\Rightarrow \quad m\left(1 - \frac{1}{1+\delta\hat{\tau}}\right) = -\frac{1}{3}\frac{m^3}{1+\delta\hat{\tau}} + \cdots$$

or $$m(\delta\hat{\tau}) \cong -\frac{1}{3}m^3$$

$$\Rightarrow \quad \boxed{m^2 \cong 3\left(\frac{\tau_c - \tau}{\tau_c}\right)}$$

so $m \sim (\tau_c - \tau)^{\frac{1}{2}}$ — as in van der Waals theory

How does magnetic susceptibility blow up?



consider slope of isotherm that is close to critical

write $m = \tanh\left(\frac{m}{\hat{\tau}} + \frac{\mu}{\tau}H\right)$

and use $$\tanh(A+B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$$

$$\Rightarrow \quad m = \frac{\tanh(m/\hat{\tau}) + h}{1 + h\tanh(m/\hat{\tau})}, \text{ where } h = \tanh\left(\frac{\mu}{\tau}H\right)$$

$$\Rightarrow \quad h = \frac{m - \tanh(m/\hat{\tau})}{1 - m\tanh(m/\hat{\tau})}$$

Now -- we use

$$\chi_\tau = \left(\frac{\partial M}{\partial H}\right)_\tau = \left(\frac{\partial M}{\partial m}\right)_\tau \left(\frac{\partial m}{\partial h}\right)_\tau \left(\frac{\partial h}{\partial H}\right)_\tau,$$ and compute $\left(\frac{\partial m}{\partial h}\right)_\tau$ for $\hat{\tau} = 1 + \delta\hat{\tau}$

10.30

Expand $\tanh x = x - \frac{1}{3}x^3$

$$\Rightarrow h = \left[m - m/\hat{\tau} + \tfrac{1}{3}\left(m/\hat{\tau}\right)^3\right]\left[1 + m^2/\hat{\tau}\right]$$

$$= m\left(1 - \frac{1}{\hat{\tau}}\right) + m^3\left(\frac{1}{3\hat{\tau}^3} + \frac{1}{\hat{\tau}}\left(1 - \frac{1}{\hat{\tau}}\right)\right)$$

$$\cong m(\delta\hat{\tau}) + \frac{1}{3}m^3$$

Hence -- for $\tau > \tau_c$, $m^2 = 0$,

We have $\frac{\partial h}{\partial m} \cong \delta\hat{\tau} \cong \frac{\tau - \tau_c}{\tau_c}$

-- for $\tau < \tau_c$, $m^2 = -3\delta\hat{\tau}$

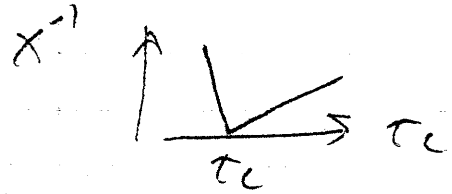
$$\frac{\partial h}{\partial m} = \delta\hat{\tau} - 3\delta\hat{\tau} = -2\delta\hat{\tau} = 2\,\frac{\tau_c - \tau}{\tau_c}$$

Also $M = n\mu m \Rightarrow \frac{\partial M}{\partial m} = n\mu$

$$h = \tanh\left(\frac{\mu}{\tau}H\right) \Rightarrow \frac{\partial h}{\partial H} \sim \mu/\tau_c$$

$$\Rightarrow \boxed{\chi_m = \frac{n\mu^2}{\tau_c} \times \begin{cases} \left(\frac{\tau - \tau_c}{\tau_c}\right)^{-1} & \tau > \tau_c \\ \frac{1}{2}\left(\frac{\tau_c - \tau}{\tau_c}\right)^{-1} & \tau < \tau_c \end{cases}}$$

(weak-field susceptibility for $\lambda = 0$)



— the same exponent and factor of 2 as in van der Waals theory

Experiment:

$$M \sim (\tau_c - \tau)^{\beta} \qquad \beta \sim .33 \qquad \text{(for "axial" magnets)}$$
$$\chi_m \sim (\tau - \tau_c)^{-\gamma} \qquad \gamma \sim 1.3$$

universal, and same as gas liquid -- But not with the mean-field exponents.

Landau theory of Phase Transitions

— A systematic and general approach to mean-field theory.

Usually, we evaluate free energy, e.g. $F(M, \tau)$ for most probable configuration (equilibrium). But we can also consider $F = U - \tau\sigma$ for other configurations (as we did implicitly in order to arrive at the maxwell construction). Then equilibrium value is found by minimizing F -- e.g.

$F(M, \tau)$ can be minimized for fixed τ to find equilibrium value M_0.

Landau's idea is that $F(M, \tau)$ is actually an analytic function that can be expanded as a power series in M, τ, and any other arguments. But the equilibrium value of F may still be nonanalytic, because the location of M_0, say, may be nonanalytic in temperature τ.

Denote by ξ the "order parameter" for the transition — which is to take the value in equilibrium

$$\xi_0 = 0 \quad \text{for } \tau \geq \tau_c$$
$$\xi_0 \neq 0 \quad \text{for } \tau < \tau_c$$

E.g. $\xi = M$ or $\xi = \rho_{liquid} - \rho_{gas}$

Called order parameter because

$\xi_0 = 0$ in a system with no "long-range order"

$\xi_0 \neq 0$ in ordered phase

(all spins favor pointing the same way)

We'll assume system has $\xi \to -\xi$ symmetry (good assumption for a magnet). This means F is an even function of ξ. So $\xi_0 = 0$ respects the symmetry, but $\xi_0 \neq 0$ "breaks" it.

There will be two states $\pm\xi_0$ with same free energy — equally good as equilibrium states

— "spontaneous symmetry breaking"

We assume F can be expanded in powers of ξ

$$F(\xi,\tau) = g_0(\tau) + \frac{1}{2} g_2(\tau)\,\xi^2 + \frac{1}{4} g_4(\tau)\,\xi^4 + \frac{1}{6} g_6(\tau)\,\xi^6 + \cdots$$

Note: Landau's theory does not really require this expansion to be convergent, it is good enough for it to be a reasonable (asymptotic) approximation carried out to e.g. order ξ^4 ---

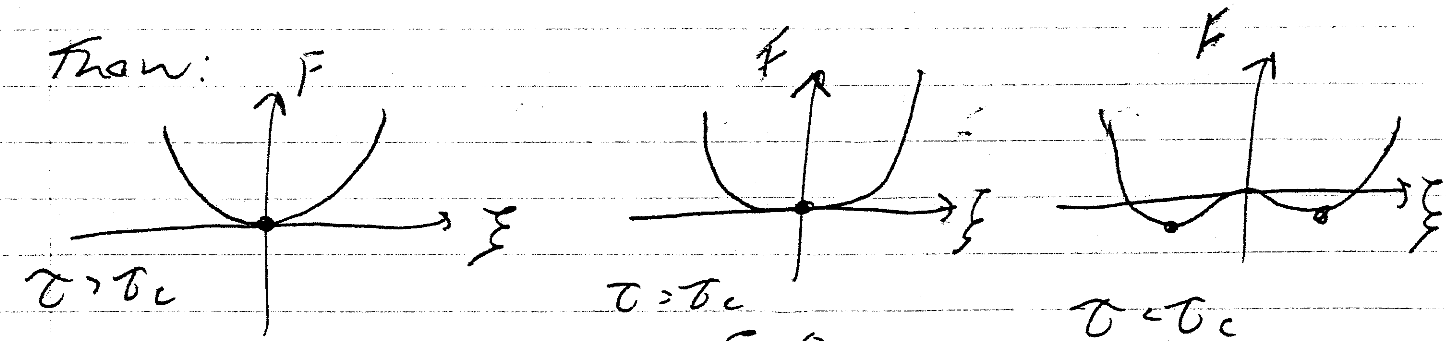
Now -- Landau says -- suppose that $g_2(\tau)$ has a zero at $\tau = \tau_c$

with $\quad g_2 > 0 \quad \tau > \tau_c$

$\qquad\quad g_2 < 0 \quad \tau < \tau_c$

suppose also that $\quad g_4(\tau_c) > 0$

Then:



$\tau > \tau_c \qquad\qquad \tau = \tau_c \qquad\qquad \tau < \tau_c$

$\xi_0 = 0 \qquad\qquad \xi_0 = 0 \qquad\qquad \xi_0 \neq 0$

The order parameter ξ turns on as we lower temperature through τ_c

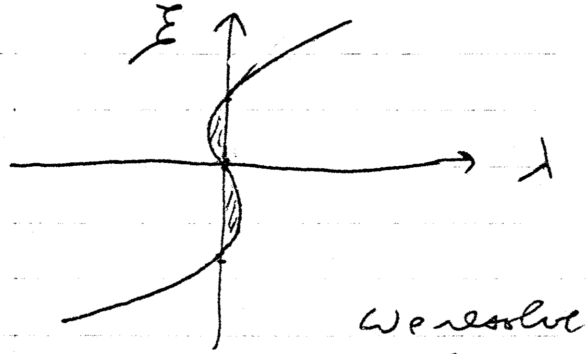
$\Longrightarrow$ phase transition

For $\tau < \tau_c$, imagine coupling an "external field" λ to ξ

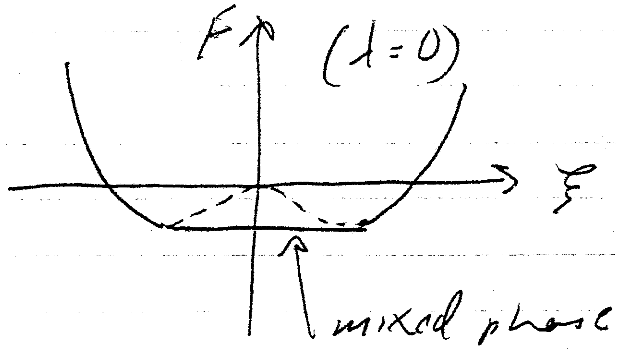
$$F(\xi,\tau) \longrightarrow F(\xi,\tau) - \lambda\xi$$

(Find $G(\lambda,\tau)$ by minimizing wrt λ)

Extremum at $\lambda = \dfrac{\partial F}{\partial \xi}$



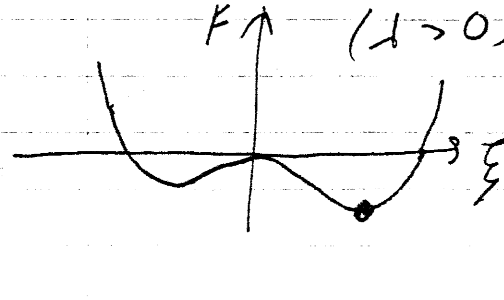
There is an instability in region where F is concave down

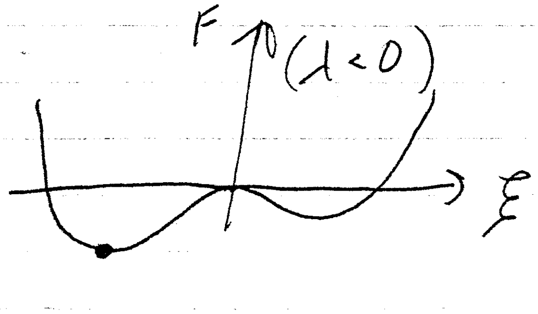


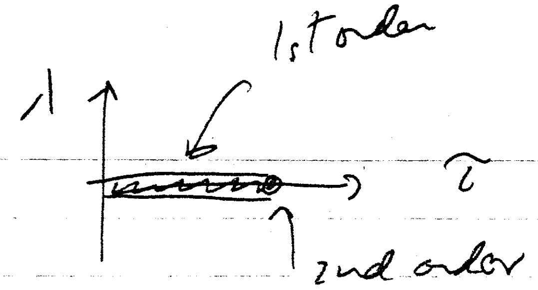
We resolve this with maxwell construction

(F is free energy of a homogeneous phase, but mixed phase has lower free energy, and is stable)

The mixed phase is favored only when external field λ is strictly zero. For $\lambda \neq 0$, one min or the other will be favored

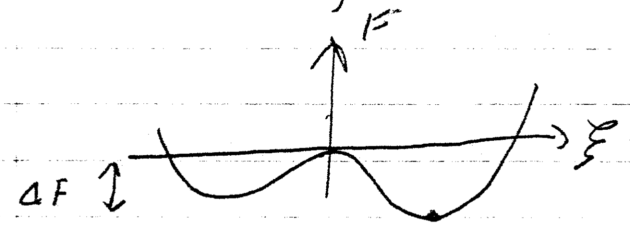






(where ξ_0 changes sign. No latent heat, however.)

There is a line of 1st order phase transitions along $\lambda = 0$, which terminates at the critical point



Note that wrong sign of ξ is metastable if external field is weak. There is a free energy barrier $\Rightarrow$ hysteresis.

Critical exponents

Onset of order parameter.

Expand $g_2(\tau)$ about its zero..

$$g_2 \sim \alpha(\tau - \tau_c) + \cdots \qquad \alpha > 0$$

$$F(\xi, \tau) \simeq g_0 + \tfrac{1}{2}\alpha(\tau - \tau_c)\xi^2 + \tfrac{1}{4} g_4 \xi^4 + \cdots$$

(Can evaluate g_0, g_4 at $\tau = \tau_c$)

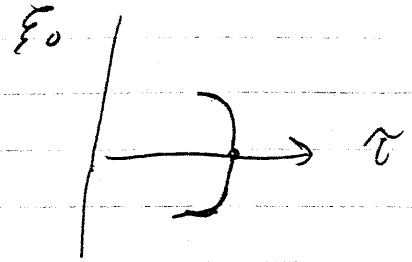
Minimize: $$\frac{\partial F}{\partial \xi} = 0 = \alpha(\tau - \tau_c)\xi + g_4 \xi^3$$

Solutions $\xi = 0$ $\qquad \xi^2 = \alpha/g_4 \, (\tau_c - \tau)$

Minimum is at

$$\xi_0 = 0 \quad , \quad \tau > \tau_c$$

$$\xi_0 = \pm\left(\frac{\alpha}{g_4}\right)^{\frac{1}{2}}(\tau_c - \tau)^{\frac{1}{2}} \quad , \quad \tau < \tau_c$$



Characteristic mean-field behavior $\quad \beta = \frac{1}{2}$

Susceptibility

Recall $\quad \lambda = \left(\frac{\partial F}{\partial \xi}\right)_\tau$

Define $\quad \chi = \left(\frac{\partial \xi}{\partial \lambda}\right)_\tau$ or $\quad \chi^{-1} = \left(\frac{\partial^2 F}{\partial \xi^2}\right)_\tau$

$$\left(\frac{\partial^2 F}{\partial \xi^2}\right)_\tau \sim \alpha(\tau - \tau_c) + 3 g_4 \xi^2 + \cdots$$

– evaluate at ξ_0

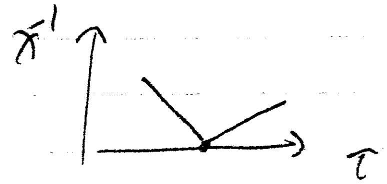
So $\quad \chi^{-1} = \alpha(\tau - \tau_c) \qquad (\tau > \tau_c)$

$$\chi^{-1} = \alpha(\tau - \tau_c) + 3 g_4 \frac{\alpha}{g_4}(\tau_c - \tau)$$

$$= 2\alpha(\tau_c - \tau) \qquad (\tau < \tau_c)$$

We find -- again $\quad \gamma = \gamma' = 1$,

and the factor of 2 difference in slope



Specific Heat

$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_\xi$ is continuous $\Rightarrow$ no latent heat

specific heat is $C_\xi = \tau\left(\frac{\partial \sigma}{\partial \tau}\right)_\xi = -\tau\left(\frac{\partial^2 F}{\partial \tau^2}\right)_\xi$

$$\frac{\partial^2 F}{\partial \tau^2} = g_0'' + g_2'' \tfrac{1}{2}\xi^2 + g_4'' \tfrac{1}{4}\xi^4$$

so $\quad C_\xi \cong -\tau \; g_0''(\tau) \qquad \tau > \tau_c$

$$C_\xi \cong -\tau\left[g_0''(\tau) + g_2''(\tau)\,\frac{1}{2}\,\frac{\alpha}{g_4}(\tau_c - \tau) + \sim\right]$$

$\tau < \tau_c$

$\cancel{\text{or } C_\xi^{(+)} > C_\xi^{(-)} \sim \frac{1}{2}\frac{\tau_c \alpha}{g_4} g_2''(\tau_c)}$

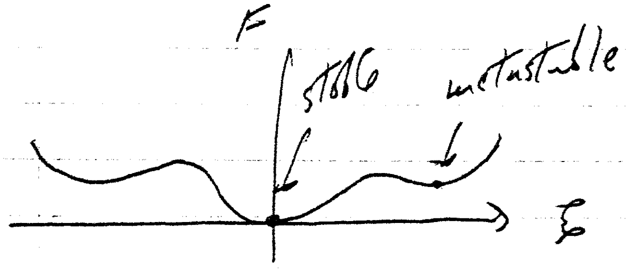
Thus C_ξ is continuous at $\tau = \tau_c$, but its 1st derivative is <u>not</u> continuous.

NOTE: critical exponent δ (critical isotherm)
$\epsilon = 0 \Rightarrow \quad h \sim \xi^3 \Rightarrow \quad \delta = 3$

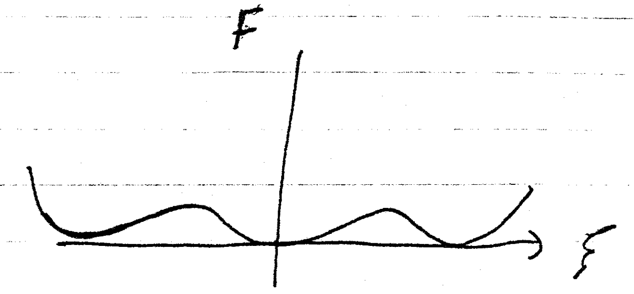
1st order Transition

In the above discussion, onset of symmetry breakdown $(\xi_0 \neq 0)$ was continuous but not analytic. The general theory can also accomodate a discontinuous onset

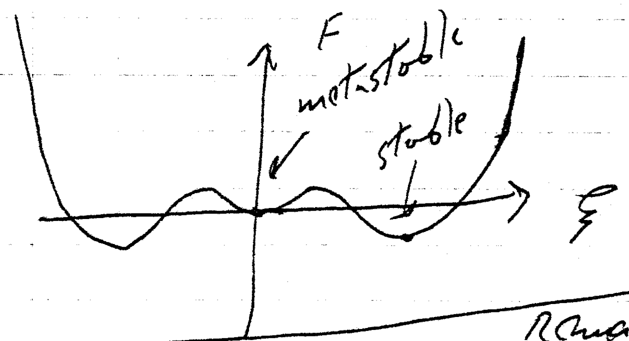
E.g. suppose $g_4(\tau_0) < 0$ but $g_6(\tau_0) > 0$
where $g_2(\tau_0) = 0$



$\tau > \tau_c$



$\tau = \tau_c$



— Order parameter jumps. There is hysteresis, and latent heat too (different values of $\sigma = -\frac{\partial F}{\partial \tau}$)

Remark: In some systems, continuous change of symmetry is not possible (e.g. solid-liquid) so transition must be 1st order
solid-liquid line cannot terminate

Scaling

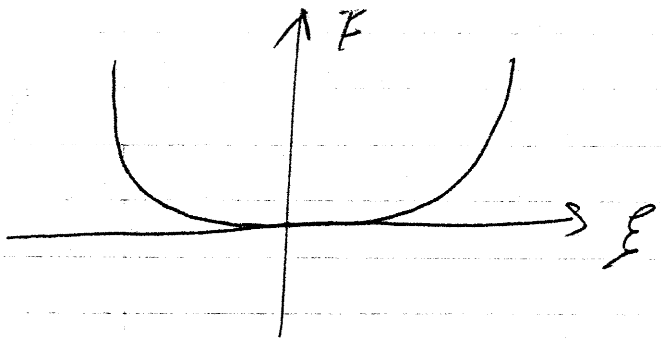
Landau theory is simple yet powerful. But it is wrong. Specific heat is not continuous — Typically, it blows up. β, γ, other exponents are wrong.

Further while there is some tendency for exponents to be universal, this is not completely true experimentally. E.g. can have different values of β, γ depending on symmetries of system

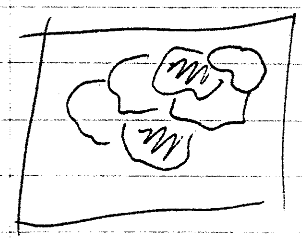
So the assumption underlying theory — that F can be expanded around critical point — is wrong.

Focus of modern theory of critical phenomena is to understand the singularities of $F(\xi, T)$ and their origin — Great progress made in late 60's and early 70's (Kadanoff, Wilson, ---)

Key feature that makes mean field approach inapplicable is that, near the critical point, there can be large, long-wavelength fluctuations away from most probable configuration.



This happen because F is very flat near $\xi = 0$ — the "restoring force" that opposes an excursion of ξ becomes weak and ineffective. (Frequency → 0, or time scale for fluctuation to relax → ∞. "critical slowing down")

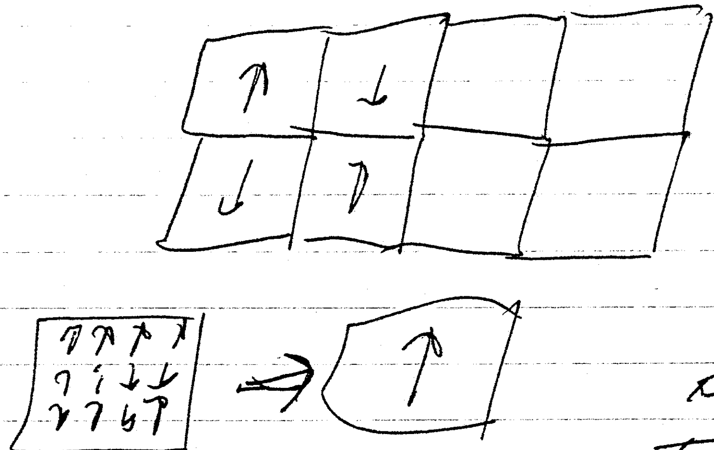


Dramatic illustration in laboratory of long wavelength fluctuations: "critical opalescence"

Fluid gets cloudy, because regions with $\delta\rho > 0$ or $\delta\rho < 0$ become comparable in size to wavelength of visible light, and so scatter the light

So — as $\tau \to \tau_c$, the fluctuations have no characteristic length scale — fluctuations on all scales are occurring — and we are faced with the problem of understanding a system in which all scales of length are equally important.

Central idea proposed by Kadanoff, systematically developed by Wilson — idea of "scaling" or "renormalization group". E.g. spins on a lattice: only near neighbors are coupled together by Hamiltonian, but distantly separated spins are in fact strongly correlated because of the long wavelength fluctuations



Describe as a system of interacting spin droplets, with nearby cells of spins coupled together

then — group cells together into even bigger cells.

Eventually — thing's scale — the bigger cells are coupled together in same way as the smaller cells (assuming both are small compared to "correlation length" which $\to \infty$ at τ_c).

E.g. $L_{\text{Big cell}} = \Omega\, L_{\text{Little cell}}$.

Then

$$F_{\text{Big Cell}}(\bar{\xi}, \overline{\delta\hat{\tau}}) = \Omega^d F_{\text{Small cell}}(\xi, \delta\hat{\tau})$$

$\delta\hat{\tau} = \dfrac{\tau - \tau_c}{\tau_c}$, d: dimensionality (normally 3) → because F is extensive

$\xi, \delta\hat{\tau}$ are renormalized

$$\bar{\xi} = \Omega^p \xi$$
$$\overline{\delta\hat{\tau}} = \Omega^q \delta\hat{\tau}$$

p, q powers that are typically noninteger

If big cells and small cells actually couple the same way, then

$$F(\Omega^p \xi, \Omega^q \delta\hat{\tau}) = \Omega^d F(\xi, \delta\hat{\tau})$$

— scaling

why a new temperature? In terms of new (larger) unit of distance... correlation length is ~~[illegible]~~ smaller ⇒ we are farther from criticality.

Dimensional analysis? Microscopic distance scale does not drop out ⇒ anamolous dimensions

The powers p, q are origin of nonanalytic behavior of F, and they can be related to critical exponents.

Often, they can be calculated.

[Ω is dimensionless ratio $\dfrac{L_{\text{Big}}}{L_{\text{Little}}}$.]

Remarks:

- Mean field theory will work if fluctuations are not so important. Philosophy of MFT is that each spin interacts with average of other ~~spins~~ – while in fact spin interacts only with nearby spins. Philosophy works better at higher dimensionality d

  There is an "upper critical dimension" at and above which the Landau prediction of exponents works. For ferromagnet

$$d_{upper} = 4$$

- Conversely – fluctuations are more important for lower d. At sufficiently low dimension, fluctuations destroy long range order at any nonzero temperature. (Lower critical dim)

$$d_{lower} = 1 \quad \text{(axial magnet)}$$
$$d_{lower} = 2 \quad \text{(rot. inv. magnet)}$$

e.g. $\underbrace{\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow}_{\text{kink}}$ In 1d, there are kinks at any finite $T \Rightarrow$ No LRO

- Landau theory is the starting point of a systematic approximation:

  $d = 4 - \epsilon$ – Calculate exponents as an expansion in ϵ, and extend to $\epsilon = 1$

• Universality

when the RG-improved Landau theory is invoked, the concept of universality survives, but in a somewhat restricted sense.
Not true that all critical phenomena have the same exponents. But one can identify "universality classes." Disparate phenomena exhibit same exponents (same scaling) — if they have the same underlying symmetries