4. Thermal Radiation and Planck Distribution

Read: Chapter 4 (May skip "electrical noise" section)

Do: Prob 1, 2, 4, 5, 6, 7, 8

The Planck Distribution

In quantum theory, light consists of particles
Einstein's light quantum hypothesis (1905)
Not assumed by Planck (1900), and not widely accepted until 1920's.

$$\epsilon_{1\,\text{photon}}(\omega) = \hbar\omega$$

$$\epsilon_{s\,\text{photons}}(\omega) = s\hbar\omega$$

The particles are "indistinguishable" i.e. there is a unique way to "put s photons in a mode"

In a (finite size) cavity, only certain (discrete) ω's are allowed (see below). For a cavity in thermal contact with reservoir (temp τ) how many photons at frequency ω?

Prob distribution

$$P_\omega(s) = \frac{1}{Z_\omega} e^{-s\hbar\omega/\tau} \quad \text{(Boltzman factor)}$$

$$Z_\omega = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau}$$

use $\sum_{s=0}^{\infty} x^s = \frac{1}{1-x} \Rightarrow Z_\omega = \frac{1}{1-e^{-\hbar\omega/\tau}}$

so — $P_\omega(s) = e^{-s\hbar\omega/\tau}\left(1-e^{-\hbar\omega/\tau}\right)$

Find expectation value of "occupation number" of photon mode with frequency ω:

$$\langle s \rangle_\omega = \sum_{s=0}^{\infty} s \, P_\omega(s)$$

Note $\displaystyle\sum_{s=0}^{\infty} s e^{-s\hbar\omega/\tau} = -\frac{d}{dy} \sum_{s=0}^{\infty} e^{-sy}$, $y = \hbar\omega/\tau$

$$= -\frac{d}{dy} (1 - e^{-y})^{-1}$$

$$= \frac{e^{-y}}{(1-e^{-y})^2}$$

and $\displaystyle\langle s \rangle_\omega = (1 - e^{-y})\left(\sum_{s=0}^{\infty} s e^{-sy}\right) = \frac{e^{-y}}{1-e^{-y}}$

$$= 1/e^{y}-1$$

or $\boxed{\langle s \rangle_\omega = \dfrac{1}{e^{\hbar\omega/\tau} - 1}}$

– the Planck distribution function.

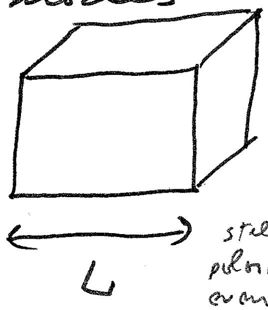
Limits:

$$\langle s \rangle_\omega = \frac{1}{e^{\hbar\omega/\tau}-1} \Rightarrow \begin{array}{ll} e^{-\hbar\omega/\tau} & \tau \to 0 \\ \tau/\hbar\omega & \tau \to \infty \end{array}$$

$$\langle \varepsilon \rangle_\omega = \hbar\omega \langle s \rangle_\omega = \frac{\hbar\omega}{e^{\hbar\omega/\tau}-1} \Rightarrow \begin{array}{l} \hbar\omega e^{-\hbar\omega/\tau}, \ \tau \to 0 \\ \tau, \ \tau \to \infty \end{array}$$

(This is the "classical" or $\hbar \to 0$ limit; typical energy stored in a mode is temperature τ – a formula with no $\hbar$ appearing)

These formulas hold for each mode of radiation in a cavity. To find e.g. total energy of photon gas in thermal equilibrium in a cavity, we must sum over the modes



E.g. suppose the cavity is a cubic box of side L, and that electric field $\vec{E}_{\parallel}$ vanishes on the cavity walls (conducting cavity)

still 2 polarizations, even if not conducting

Solve wave equation $\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$ by expanding in normal modes (Fourier analysis) $\Rightarrow \omega^2 = c^2 \vec{K}^2$

e.g.
$$E_x = E_x^{(0)} \cos(K_x x)\sin(K_y y)\sin(K_z z)e^{i\omega t}$$
$$E_y = E_y^{(0)} \sin(K_x x)\cos(K_y y)\sin(K_z z)e^{i\omega t}$$
$$E_z = E_z^{(0)} \sin(K_x x)\sin(K_y y)\cos(K_z z)e^{i\omega t}$$

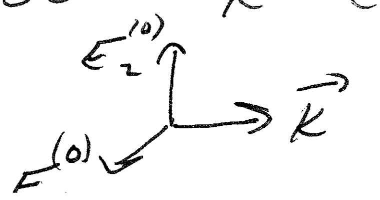
(Need these forms to satisfy $\vec{\nabla}\cdot\vec{E} = 0$, see below

then
$$K_x L = \pi n_x$$
$$K_y L = \pi n_y$$
$$K_z L = \pi n_z$$

$n_{x,y,z}$ are integers > 0

This solves wave eqn for $\omega^2 = c^2(K_x^2 + K_y^2 + K_z^2) = c^2 \vec{K}^2$

In addition, we must have
$$\vec{\nabla}\cdot\vec{E} = 0 \quad \text{or} \quad \vec{K}\cdot\vec{E}^{(0)} = 0$$



2 polarizations

Now since

$$\vec{K} = \pi/L \,(n_x, n_y, n_z)$$

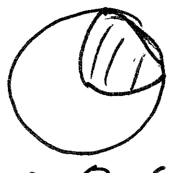
the sum over allowed values of $\vec{K}$ can be replaced by an integral as $L \to \infty$

$$2 \sum_{n_{x,y,z}} \to 2\left(\frac{L}{\pi}\right)^3 \int_0^\infty dK_x \int_0^\infty dK_y \int_0^\infty dK_z$$

(from counting polarizations)

And frequency $\omega^2 = c^2 \vec{K}^2$ depends only on length of $\vec{K}$, so

$$\int d^3K \longrightarrow \frac{1}{8} 4\pi \int_0^\infty dK\, K^2$$



($K_{x,y,z} > 0$, so integral over $\frac{1}{8}$ of sphere)

so $$\sum_{\vec{n}} = 2\left(\frac{L}{\pi}\right)^3 \frac{1}{8} 4\pi \int_0^\infty dK\, K^2$$

and e.g. $$U = \sum_{\vec{n},\, pol} \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1}$$

$$= L^3/\pi^2 \int dK\, K^2 \frac{\hbar c K}{e^{\hbar c K/\tau} - 1}$$

Let $x = \hbar c K/\tau$

$$= L^3/\pi^2 \left(\frac{\tau}{\hbar c}\right)^4 \hbar c \underbrace{\int_0^\infty dx \frac{x^3}{e^x - 1}}_{\text{dimensionless integral} \;=\; \pi^4/15}$$

Volume $V = L^3$

$$\boxed{U/V = \frac{\pi^2}{15} \frac{1}{(\hbar c)^3} \tau^4}$$

Stefan-Boltzmann radiation law

Lecture #8 (16 April 91)

can express as frequency integral of spectral density

$$\boxed{U/V = \int d\omega \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/\tau} - 1}}$$

↗ "spectral density" u_ω

This is the Planck "black-body" spectrum of radiation

(Peak at $x = \hbar\omega/\tau = 2.82$)

$\hbar = 6.582 \times 10^{-16}$ eV sec
$c = 2.9979 \times 10^{10}$ cm sec^{-1}
$K = 11605$ °K/eV
$\Rightarrow$ $\tau = 2.725$°K
$\omega = 356.7 \times 10^9$ Hz $\hbar\omega \simeq 2.348 \times 10^{-4}$ eV
$\omega_{peak} = 1006.0 \times 10^9$
$f_{peak} = 160.2 \times 10^9$
$\lambda_{peak} = c/f = 1.87$ mm
"microwave"

So cosmic: $\tau/k = 2.7$°K ← (cobe value) 2.73 °K, 2.725 ± 0.002

or sun: $\tau/k \sim 6000$° actually 5785°K

In the low frequency limit, the spectral density behaves like

$$u_\omega \simeq \frac{\omega^2}{\pi^2 c^3} \tau \qquad (\omega \to 0)$$

For sun
Peak is 880 nm
~~Mean is 312 nm~~
Yellow is 570-590 nm

This is "Rayleigh-Jeans Law" - see below
Purely classical (no $\hbar$) - If it persisted for all ω,

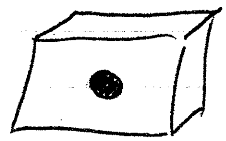
sun should be white hot but shift to red by scatter (which makes sky blue

We would have

$$U/V = \int d\omega\, u_\omega = \infty$$

Quantum physics intervenes to prevent this ultraviolet catastrophe.

Black Body

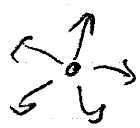


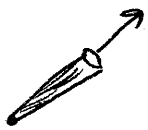
Suppose we cut a small hole in a cavity. At low temperature, the hole is black: radiation incident on it is absorbed by the cavity. If cavity is hot, radiation is emitted through the hole. $\hbar\omega_{vis} \sim 1\,eV \sim 10^4\,{}^\circ K$

$T = 300^\circ K \Rightarrow kT \sim .0258\, eV \sim \frac{1}{38.7}\, eV$ $\qquad e^{-30} \sim 10^{-13}$

What is radiation energy flux (rate of emission per unit time and area)?

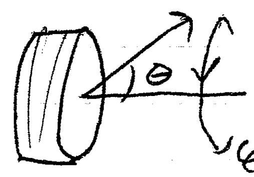
Radiation gas is isotropic — photon velocities are uniformly distributed in solid angle (and speed = c at all frequencies)





i.e. energy density due to photons with velocity in direction of solid angle $d\Omega$ is

$$U/V \, \frac{d\Omega}{4\pi}$$



In time interval dt, quanta are emitted by the hole if within $c(dt)\cos\theta$ of edge of the cavity

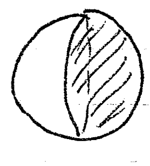
Energy due to such quanta is

$$d(\text{Energy}) = U/V\,(\text{Area})\, c(dt)\cos\theta \;\tfrac{1}{2}\, d\cos\theta$$

$$J \equiv \text{Flux} = \frac{\text{Energy}}{\text{Area}\cdot(dt)} = \frac{U}{V} c \int_0^1 \frac{1}{2} d\cos\theta \, \cos\theta$$

(only ones moving toward hole are counted)

$$= \frac{U}{V} \frac{1}{4} c$$



integrate direction of photons over hemisphere

So we have

$$J = \underbrace{\frac{\pi^2}{60} \frac{1}{\hbar^3 c^2}}_{\sigma_B} \tau^4$$

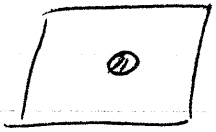
σ_B - the Stefan-Boltzmann constant

Rate at which energy is lost through a hole in a cavity - in equilibrium with radiation at temperature τ (per unit area)

A black-body is one that absorbs all radiation incident on its surface (does not reflect)

Any black body radiates like a hole in a cavity at the same temperature

= "thermal contact" of cavity and body



To see this - imagine closing the hole by covering it with surface of the black body. If cavity and body are in thermal equilibrium at temp τ, then body is emitting at the same rate that it is absorbing. - It emits radiation into the cavity at the same rate as radiation is escaping through the hole

Non-black body

Absorption = $a \cdot$ (black body absorption)
a = "absorptivity"

Emission = $e \cdot$ (black body emission)
e = "emissivity"

claim: $a = e$

why? Consider such a body at temp τ in equilibrium with radiation in a cavity. It must then emit and absorb at the same rate

We could also cover the body with a filter that transmits radiation in frequency range $(\omega, \omega + d\omega)$ and omits all else then see that

$$a(\omega) = e(\omega)$$

("thermal contact in a frequency interval")

— Kirchoff's Law

Historical Notes (see "Subtle is the Lord" by Abraham Pais.

Kirchoff (1859) was the first to realize that the spectral density of radiation in equilibrium with a body has universal character

$$U/V = \int d\omega \; a(\omega) \, u_\omega$$

? Why $a(\omega)$ in U/V of radiation?

where $u_\omega(\tau)$ is a universal function independent of the constitution of the body.

So it was recognized that this poses a fundamental problem — what is this function $u_\omega(\tau)$?

Stefan conjectured in 1879, and Boltzman proved in 1884 that for a black body

$$U/V = \int d\omega\, u_\omega(\tau) = (\text{const}) \cdot \tau^4$$

Proof uses thermodynamics and maxwell radiation theory. Challenge — what is this constant of proportionality?

In 1893, Wien extended these arguments to show

$$u_\omega(\tau) = \omega^3 f(\omega/\tau)$$

since ω/τ is not dimensionless, depends on a physical constant with dimensions

— as far as one could go with classical thermodynamics.

Became an experimental priority to measure $u_\omega(\tau)$. (Need to measure for infrared — an experimental challenge)

Rubens-Kurlbaum present their data to Prussian academy of sciences on Oct 25, 1900 — fit by a curve found by Planck

$$u_\omega(\tau) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/\tau} - 1}$$

Planck radiation law

(The exponential cutoff for large ω had been anticipated by Wien in 1896.)