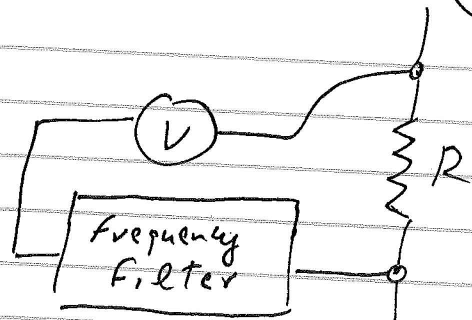
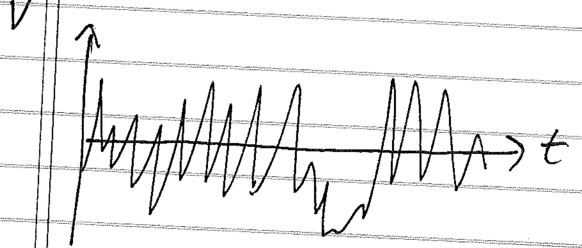
Johnson-Nyquist Noise

At nonzero temperature, there are voltage fluctuations in a resistor, called



Johnson-Nyquist noise (discovered by Johnson and explained by Nyquist).

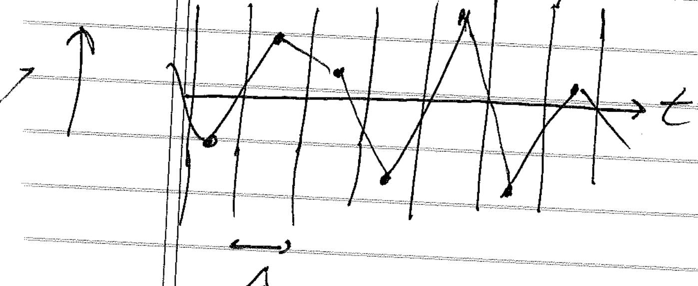


Suppose we take traces of the noise, sampling it many times. We Fourier analyze each trace, then filter, including only the Fourier modes in a narrow "band" of width df in frequency space. Averaging this filtered noise at a particular time t over the many samples, we find

$$\langle V(t) \rangle = 0 \qquad \langle [V(t)]^2_{\text{filtered}} \rangle = 4\tau R\, df$$

where τ is temperature, R is resistance, and df is bandwidth. We say this electrical noise is "white noise" because, like white visible light, all frequencies occur with equal weight.

Another way to describe white noise is this:



Consider the average voltage over a short time interval of width Δ.

Suppose that we generate voltage traces by sampling independently in each time interval from a distribution with

$$\langle V \rangle = 0 \qquad \langle V^2 \rangle = \bar{V}^2$$

(there are no correlations in the noise from one time window to the next). Suppose we record the voltage in each time window for total time T, so there are $N = T/\Delta$ values recorded, denoted

$$\{V_n,\ n = 0, 1, 2, \ldots, N-1\}.$$

Fourier transforming we may write

$$V_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{-2\pi i m n/N}\, \tilde{V}_m \quad \text{, where}$$

$$\tilde{V}_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{2\pi i m n/N}\, V_n$$

In terms of time $t_n = n\Delta$ and frequency $f_m = \frac{m}{T} = \frac{n}{N\Delta}$ we may write

$$\exp(-2\pi i m n/N) = \exp(-2\pi i f_m t_n).$$

If we integrate the voltage for time T

$$\int_0^T V\,dt \approx \sum_{n=0}^{N-1} \Delta V_n$$

and
$$\left\langle \left(\int_0^T V\,dt\right)^2 \right\rangle = \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} \Delta^2 \underbrace{\langle V_n V_{n'} \rangle}_{\delta_{nn'} \bar{V}^2} = N\Delta^2 \bar{V}^2$$

$$= (N\Delta)(\Delta \bar{V}^2) = T(\Delta \bar{V}^2)$$

Thus the average voltage has mean square fluctuations

$$\left\langle \left(\frac{1}{T}\int_0^T V\,dt\right)^2 \right\rangle^{\frac{1}{2}} = \left(\frac{1}{T}\Delta \bar{V}^2\right)^{\frac{1}{2}}$$

which decreases like $1/\sqrt{T}$. To detect a nonzero static voltage signal, we should integrate for a time

$$T_0 \simeq \frac{\Delta \bar{V}^2}{V_0} \qquad \text{where } V_0 \text{ is the signal;}$$

the quantity $\Delta \bar{V}^2$ characterizes the noise, and has a smooth limit as the time window Δ gets small.

To see that the fluctuations have constant power per unit of bandwidth, consider

$$\tilde{V}_m \tilde{V}_{m'}^* = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} e^{2\pi i m n/N} e^{-2\pi i m' n'/N} \underbrace{\langle V_n V_{n'} \rangle}_{\delta_{nn'} \bar{V}^2}$$

$$= \delta_{mm'} \bar{V}^2$$

The noise in Fourier modes is also uncorrelated.

Now if we consider the filtered noise at a particular time, where we sum over only some of the Fourier modes:

$$\langle [V_n^2]_{filtered} \rangle = \frac{1}{N} \sum_{modes} \bar{V}^2 = \frac{1}{N} (\#\text{ of modes})\, \bar{V}^2$$

where "# of modes" means the number of modes that the filter allows to pass. Since

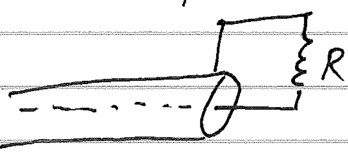
$f_m = \frac{m}{T}$, the number of modes dm in frequency band df is $dm = T(df) = N\Delta\, df$

$$\Rightarrow \langle [V^2]_{filtered} \rangle = \underbrace{(\Delta \bar{V}^2)}\, df,$$ which is white noise.

The noise strength is expressed in units

Volts $\cdot$ sec$^{\frac{1}{2}}$ or Volts$/\sqrt{Hz}$.

To derive a formula for the noise strength, we rely on some electrical engineering lore:



A transmission line (like a coaxial cable) can be "impedance matched" to a resistor R so that all power incident on the resistor is absorbed; when matched, the transmission line has the same impedance as the resistor. Thus the resistor is in effect a "black body" — in thermal equilibrium it radiates at the same rate it absorbs, so the rate at which the resistor dissipates power must agree with the rate at which the transmission line delivers power to the resistor.

Now let's compute that power. For a transmission line of length L, the allowed wave numbers are

$$k_n = \frac{\pi}{L} n \quad \text{where } n \text{ is a positive integer,}$$

and frequencies are $\omega_n = v k_n = \frac{\pi v}{L} n$

where v is propagation speed. Energy stored in mode with frequency ω is

$$\langle \mathcal{E}(\omega) \rangle = \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1} \simeq \tau \quad \text{for } \hbar\omega \ll \tau$$

The energy per unit length in the line is

$$U/L = \frac{\tau}{L} \sum_n = \frac{\tau}{\pi v} \int d\omega = \frac{2\tau}{v} \int df \quad \text{where } f = \frac{\omega}{2\pi}$$

Energy is emitted from the line at rate $(U/L)v$, with half that power emitted at each end.

So power emitted by one end is

$$\text{Power} = \tau\, df \quad \text{in the bandwidth interval } df.$$

The power dissipated by the resistor must therefore be

$$\text{Power} = R\langle [I^2]_{\text{filtered}} \rangle = \tau\, df$$

and since the total impedance of the resistor and line is $2R$, we have $V = 2IR \Rightarrow$

$$\frac{\langle [V^2]_{\text{filtered}} \rangle}{4R} = \tau\, df$$

or $$\boxed{\langle [V^2]_{\text{filtered}} \rangle = 4R\tau\, df}$$

This is Nyquist's formula for Johnson noise.

In order of magnitude, using $K_B \simeq 1.4 \times 10^{-23}\ J/{}^\circ K$,

the noise is about $1\,\mu V$ for $R = 50\Omega$, and MHz bandwidth, at $T = 300^\circ K$

$$\langle [V^2]_{\text{filtered}} \rangle^{\frac{1}{2}} \simeq 2\left[50 \times (1.4\times10^{-23}) \times 300 \times 10^{6}\right]^{\frac{1}{2}} \simeq 10^{-6}\, V$$

(We need to integrate for $\sim 1\mu s$ to detect $1\mu V$ signal in 50Ω circuit at room temperature.)

Note that we assumed
$\hbar\omega = hf \ll K_B T$

$f \simeq 1\,\text{GHz}$ corresponds to $T \simeq \dfrac{6\times10^{-34}\times10^{9}}{1.4\times10^{-23}} \simeq 4\times10^{-2}\,{}^\circ K$

or $40\, mK$

In fact, over the past decade, circuits with $f \simeq$ few $\times$ GHz have been studied at $T \sim 20^\circ mK$ + the "quantum noise" regime.