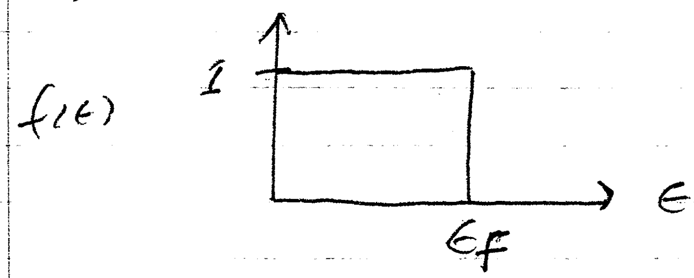
7 Fermi and Bose Gases

Now, continue to assume that gas is ideal (noninteracting), but dispense with assumption of low density $n \ll n_Q$. We now allow occupation number of orbital to be order one (or even much larger, in case of Bose gas).

These quantum gases behaves much different than classical, ideal gas. (And behave much different than each other.)

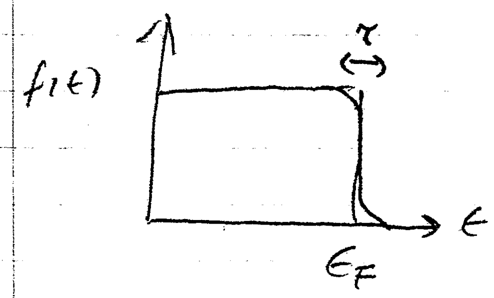
Consider $\tau = 0 \Rightarrow$ system is in ground state ("Degenerate gas")

Fermi gas has distribution function



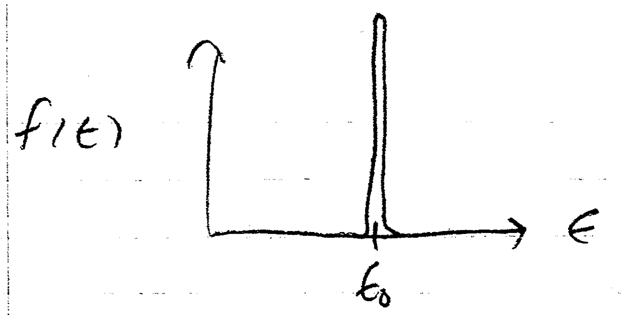
orbitals with $\epsilon < \epsilon_F$ are fully occupied, and those with $\epsilon > \epsilon_F$ are empty

If we heat up system a little ($0 < \tau \ll \epsilon_F$)



Then only those particles in orbitals close to ϵ_F can get excited.

A degenerate Bose gas has distribution function



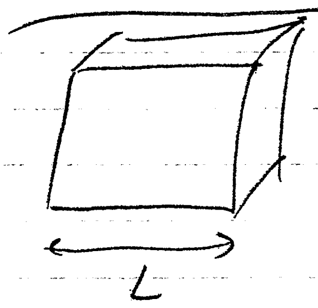
Every particle occupies the lowest orbital at $\epsilon = \epsilon_0$

When we heat system up a little, most particles want to stick together in the lowest orbital

(Here, unlike gas of a photon gas, we assume that number N of particles is conserved; doesn't change when τ changes.)

We can understand a surprising amount of real physics by studying these quantum ideal gases.

Fermi Gas



Recall, again, counting of orbitals for a single particle in a box (cubic box with side L)

Allow K: $\vec{K} = \frac{\pi}{L}(n_1, n_2, n_3)$

$n_{1,2,3}$ are integers (positive)

For L large, replace sum over n by integral

$$\sum_{\vec{n}} \rightarrow \left(\frac{L}{\pi}\right)^3 \frac{1}{8} \int d^3K$$

[compensates for overcounting, if we allow components of $\vec{K}$ to have either signs)

Enumeration of States

Since counting of states does not depend on boundary conditions, consider periodic boundary conditions. Then

$$k_x = \frac{2\pi}{L} n_x \quad , \quad k_y = \frac{2\pi}{L} n_y \quad , \quad k_z = \frac{2\pi}{L} n_z \, ,$$

where now n_x, n_y, n_z are integers that need not be positive. Approximating sum by an integral

$$\sum_{n_x} \longrightarrow \frac{L}{2\pi} \int_{-\infty}^{\infty} dk_x \, , \quad \sum_{n_y} \longrightarrow \frac{L}{2\pi} \int_{-\infty}^{\infty} dk_y \, , \quad \sum_{n_z} \longrightarrow \frac{L}{2\pi} \int_{-\infty}^{\infty} dk_z$$

Thus $\displaystyle \sum_{\text{states}} \longrightarrow \frac{L^3}{(2\pi)^3} \int d^3k$ (in 3 spatial dimensions)

Writing $\vec{p} = \hbar \vec{k}$, this becomes

$$dn = \frac{V}{(2\pi\hbar)^3} d^3p$$

where V is spatial volume, and n is number of states.

In D dimensions, this would become $dn = \dfrac{V}{(2\pi\hbar)^D} d^Dp$

Noting that $2\pi\hbar = h$, we may say that $1/h^D$ is the density of states per unit of phase-space volume $d^3x \, d^3p$.

This criterion may remind you of the Born-Sommerfeld quantization condition from Ph 12b; e.g., for periodic motion in one-dimension

$$\text{Area} = \oint p \, dq = n h \, .$$

(The phase space area enclosed by the orbit, in units of h, determines the excitation level of the quantized motion.)

or $\quad \sum_{\vec{n}} \rightarrow \frac{V}{(2\pi)^3} \int d^3K$

since momentum is $p = \hbar K$, can also write

$$\sum_{\vec{n}} \longrightarrow \frac{1}{(2\pi\hbar)^3} V \int d^3p$$

— so there is one orbital for each $(2\pi\hbar)^3 = h^3$ of phase space volume ("phase space" = $(\vec{x}, \vec{p})$ space)

[or h^D in D dimensions...] — this is easy to remember.

Degenerate electrons: a different meaning of "degenerate" from previously

For electrons there is also factor of 2 for spin degeneracy (they are spin-$\frac{1}{2}$). Find ground state energy U_0 by filling all orbitals up to ϵ_F

$$N = 2 \frac{V}{(2\pi)^3} \int_0^{K_F} 4\pi K^2 dK = \frac{V}{\pi^2} \frac{1}{3} K_F^3$$

or $\boxed{K_F^3 = 3\pi^2 \frac{N}{V} \equiv 3\pi^2 n}$ (so $\frac{2\pi}{K_F}$ comparable to interparticle spacing)

For a nonrelativistic particle $\quad \epsilon = \frac{p^2}{2m} = \frac{\hbar^2}{2m} K^2$

so $\boxed{\epsilon_F = \frac{\hbar^2}{2m} K_F^2 = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}}$

We can also find ground state energy:

$$U_0 = 2 \frac{V}{(2\pi)^3} \int_0^{K_F} 4\pi K^2 dK \frac{\hbar^2}{2m} K^2$$

$$= \frac{V}{\pi^2} \frac{\hbar^2}{2m} \frac{1}{5} K_F^5 = \frac{V}{\pi^2} \frac{1}{5} \left(3\pi^2 \frac{N}{V}\right) \frac{\hbar^2}{2m} K_F^2 = \frac{3}{5} N \epsilon_F$$

$\boxed{U_0/N = \frac{3}{5}\epsilon_F}$ Average energy per particle is 3/5 of Fermi energy

Finite τ

We'll compute expectation values as ? Not a normalized prob distribution

$$\langle X \rangle = \sum_{\vec{n}} f(\epsilon_{\vec{n}}, \tau, \mu) X_{\vec{n}}$$ Adding contrib from each orbital

$$\rightarrow \int d\epsilon\, \mathcal{D}(\epsilon) f(\epsilon) X(\epsilon)$$ FD distribution

convenient to introduce density of states per unit energy

Recall $\sum_{\vec{n}} \rightarrow \frac{V}{(2\pi\hbar)^3}\int d^3p$ (neglecting spin degeneracy)

$$p^2 = 2m\epsilon \Rightarrow dp = \tfrac{1}{2}(2m)^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}d\epsilon$$

$$\frac{V}{(2\pi\hbar)^3} 4\pi p^2 dp = \frac{V}{4\pi^2\hbar^3}(2m)^{3/2}\epsilon^{\frac{1}{2}}d\epsilon$$

so $$\boxed{\mathcal{D}(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{\frac{1}{2}} \,(\times \text{ spin factor})}$$

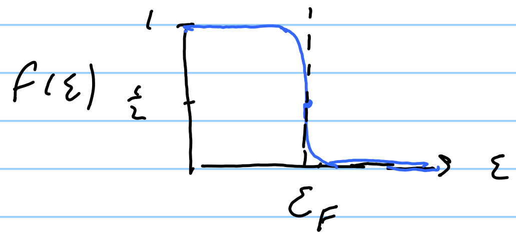
(= 2, for electrons)

Note that $f(\epsilon) = \frac{1}{e^x+1}$ where $x = (\epsilon-\mu)/\tau$

$$f - \tfrac{1}{2} = \tfrac{1}{2}\frac{e^x-1}{e^x+1} = \tfrac{1}{2}\tanh\tfrac{x}{2} \Rightarrow f(\epsilon) = \tfrac{1}{2} + \tfrac{1}{2}\tanh\left(\frac{\epsilon-\mu}{2\tau}\right)$$

hyperbolic Tangent

Heat Capacity of a nearly degenerate electron gas



For electrons in a typical metal,

$\varepsilon_F \sim$ few eV $\sim$ few $\times 10^4$ °K

while $kT \simeq \frac{1}{40}$ eV $\simeq 300$°K

at room temperature.

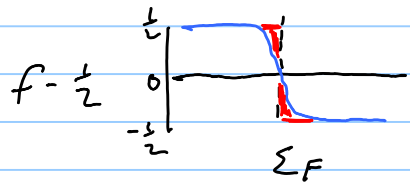
Thus, $f(\varepsilon)$ is close to 1 for $-\frac{\varepsilon-\varepsilon_F}{\tau} \gg 1$. (Only electrons close to Fermi level can be excited.)

$f(\varepsilon)$ is close to 0 for $\frac{\varepsilon-\varepsilon_F}{\tau} \gg 1$.

where $f(\varepsilon) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{2}\right)$, $\quad x = \frac{\varepsilon-\mu}{\tau} \simeq \frac{\varepsilon-\varepsilon_F}{\tau}$

We may compute the heat capacity $\frac{dU}{d\tau}$ of the nearly degenerate gas as a power series in the temperature τ, keeping only the linear term. When the temp is very small, a few electrons are removed from orbitals just below ε_F and placed in orbitals just above ε_F.

We may make the approximation that $D(\varepsilon) \cong D(\varepsilon_F)$ is an ε-independent constant in the region where $f(\varepsilon)$ deviates significantly from either $f=0$ or $f=1$.



Then since $f-\frac{1}{2}$ is an odd function about $\varepsilon=\varepsilon_F$, the shaded red areas above and below $f(\varepsilon)$ match.

This means that the total particle number N stays fixed if we turn on a small nonzero temp., and keep the chemical potential $\mu = \varepsilon_F$ fixed. Thus, 7.6

$$\left.\frac{d\mu}{d\tau}\right|_{\tau=0} = 0$$

and $0 = \left.\frac{dN}{d\tau}\right|_{\tau=0} = \frac{d}{d\tau}\int d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon,\mu,\tau)$

$$\cong \mathcal{D}(\varepsilon_F)\int d\varepsilon \frac{\partial}{\partial\tau} f(\varepsilon,\varepsilon_F,\tau)$$

– We can pull $\mathcal{D}(\varepsilon_F)$ outside the integral because $\frac{\partial}{\partial\tau} f(\varepsilon,\varepsilon_F,\tau)$ is nonvanishing only for ε near ε_F, and we ignore the τ dependence of μ because it is constant up to linear order. The integral vanishes because $\frac{\partial}{\partial\tau} f(\varepsilon,\varepsilon_F,\tau)$ is an odd function of $\varepsilon - \varepsilon_F$. To see this explicitly:

$$f(\varepsilon,\varepsilon_F,\tau) = \tfrac{1}{2} - \tfrac{1}{2}\tanh\left(\tfrac{x}{2}\right), \quad x = \frac{\varepsilon-\varepsilon_F}{\tau}$$

$$\Longrightarrow \quad \frac{\partial f}{\partial\tau} = \underbrace{\left(\frac{\varepsilon-\varepsilon_F}{\tau^2}\right)}_{\text{odd in } (\varepsilon-\varepsilon_F)} \underbrace{\tfrac{1}{4}\operatorname{sech}^2\left(\tfrac{x}{2}\right)}_{\text{even in } (\varepsilon-\varepsilon_F)}$$

Heat capacity is

$$C = \frac{dU}{d\tau} = \int d\varepsilon\, \mathcal{D}(\varepsilon)\, \varepsilon \frac{\partial}{\partial\tau} f(\varepsilon,\mu,\tau)$$

$$\cong \mathcal{D}(\varepsilon_F)\int d\varepsilon \left((\varepsilon-\varepsilon_F) + \varepsilon_F\right) \frac{\partial}{\partial\tau} f(\varepsilon,\varepsilon_F,\tau)$$

becomes integral of the even function $\frac{x^2}{4}\operatorname{sech}^2\left(\frac{x}{2}\right)$

vanishes, because integral of odd function

Therefore, $C = \mathcal{D}(\varepsilon_F)\,\tau \int_{-\infty}^{\infty} dx\; \tfrac{1}{4} x^2 \operatorname{sech}^2\left(\tfrac{x}{2}\right) = \frac{\pi^2}{3}\tau\, \mathcal{D}(\varepsilon_F).$

We may also express the density of states at the Fermi level in terms of the total number of electrons N and the Fermi energy ε_F.

We recall that $\mathcal{D}(\varepsilon) \propto \varepsilon^{1/2}$, and therefore

$N(\varepsilon_F) = \int_0^{\varepsilon_F} d\varepsilon\, \mathcal{D}(\varepsilon) \propto \varepsilon_F^{3/2}$. Since

$$\mathcal{D}(\varepsilon_F) = \frac{d}{d\varepsilon_F} N(\varepsilon_F) = \frac{3}{2}\frac{1}{\varepsilon_F} N(\varepsilon_F),$$

our expression for heat capacity becomes

$$C = \frac{\pi^2}{3}\tau\, \mathcal{D}(\varepsilon_F) = \frac{\pi^2}{2}(\tau/\varepsilon_F)\, N.$$

That makes sense --- the heat capacity is suppressed by the factor $\tau\mathcal{D}(\varepsilon_F)$ because only orbitals in an interval with width of order τ around ε_F contribute to the temperature dependence of the energy. The number of such orbitals is $\sim \tau\mathcal{D}(\varepsilon_F)$

Electrons in Metals

- treat electrons as ideal gas (!?) – see below

What is Fermi energy? Consider e.g. alkali metal; one electron per atom contributes to conduction.

$N_{electron} = N_{atom}$ – calculate from density and atomic weight

e.g. Potassium (K) $n = 1.3 \times 10^{22}\ cm^{-3}$

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = 2.1\,eV = 2.4\times 10^4\ {}^\circ K$$

(cf Rydberg)

So – K at room temperature has $\tau/\epsilon_F \sim .01$, and is very nearly degenerate

Predict:

$$C_V = \gamma\tau + A\tau^3 + \cdots$$

(Higher order terms coming from the nearly degenerate Fermi gas contribution neglected?)

degen electron gas, $\gamma = \frac{\pi^2}{2} N(\epsilon_F)^{-1}$

Debye phonon gas $A = \frac{12\pi^4}{5} N \Theta^{-3}$

At sufficiently low τ, the electron gas dominates. Where does crossover occur?

$$N \frac{\pi^2}{2}\frac{\tau}{\epsilon_F} \sim \frac{12\pi^4}{5} N \tau^3/\Theta^3 \Rightarrow (\tau/\Theta)^2 \sim \frac{5}{24\pi^2}\,\Theta/\epsilon_F$$

This typically gives $\tau \sim 1^\circ K$

E.g. Potassium $\Theta = 91^\circ K$, $\tau/\Theta \sim .008$, $\tau \sim 0.8^\circ K$

Experimentally, the $C_V \sim \gamma\tau$ law works remarkably well at low τ, but with rescaled γ — E.g., for Potassium $\gamma/\gamma_0 = 1.23$, where $\gamma_0 = \frac{N\pi^2}{2}\epsilon_F^{-1}$, the ideal gas prediction

That naive ideal gas model does not predict γ to better than 20% is no surprise. After all, gas is not ideal. Electrons have coulomb repulsion, and interact with lattice ions. Surprise is — $C_V \propto \tau$ works, with slight rescaling.

Key concept — "dressed" electron = "quasiparticle" (Landau-Fermi liquid) interactions "renormalize" electron mass etc — but dressed particles interact weakly

White Dwarfs

End point of stellar evolution — supported against collapse by electron degeneracy pressure

E.g. Sirius B (companion of Sirius)

From orbit, find $M_\odot \sim 2\times10^{33}$ g

From luminosity and temperature $L = \sigma T^4 \cdot \text{Area}$

i.e. wobbled sirius and radius of orbit of its "pup"

Find $R \sim 2\times10^9$ cm $\sim$ 20,000 Km

(Much less than $R_\odot \sim 7\times10^5$ Km)

Star is at very high density

E.g. if Hydrogen, would have interparticle spacing $\sim .01$ Å
squeezing ionizes the hydrogen

<u>Mean</u> density (of course, density not really uniform)

$$\rho \sim M/V \sim 7\times10^{4}\ \mathrm{g\,cm^{-3}} \Rightarrow \epsilon_F \sim 3\times10^{5}\,\mathrm{eV}$$
$$\sim 3\times10^{9}\,{}^\circ K$$

($\gg$ Rydberg $\Rightarrow$ H ionizes) (cf $mc^2 \sim 5\times10^{5}$ eV)

Even in core, $\tau \sim 10^{7}\,{}^\circ K \Rightarrow$ <u>highly</u> degenerate electron gas.

What <u>determines</u> this size?

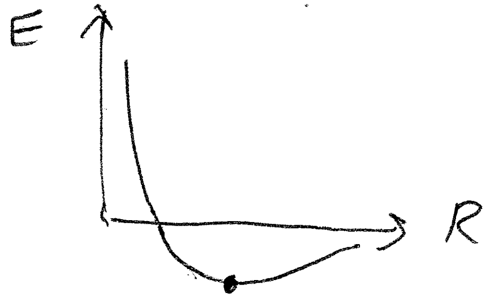
Crude estimate (we won't worry about actual variation of density with depth, so we won't try to keep numerical factors.)

Gravitational Potential energy $\sim -GM^2/R$

Electron Kinetic energy $\sim N\dfrac{\hbar^2}{m_e}\left(\dfrac{N}{V}\right)^{2/3} \cong \dfrac{\hbar^2}{m_e}\dfrac{N^{5/3}}{R^2}$

or $M \sim N m_p$ (m_p = proton mass)

$$\text{Kinetic} + \text{Potential} = \frac{\hbar^2}{m_e}\left(\frac{M}{m_p}\right)^{5/3}\frac{1}{R^2} - \frac{GM^2}{R}$$



star chooses R to minimize energy

$$R \sim \frac{1}{GM^2}\left(\frac{\hbar^2}{m_e}\right)\left(\frac{M}{m_p}\right)^{5/3}$$

$$\boxed{R \cong \frac{\hbar^2}{G m_e m_p^{5/3}} M^{-1/3}}$$

So a heavier star is actually smaller, and mean density $M/V \sim M^2$

Numerically $R \sim 10^{20} g^{\frac{1}{3}} cm \, M^{-1/3}$

or $M \sim 10^{33} g \Rightarrow R \sim 10^{9} cm \sim 10^{4} Km$

(size comparable to earth)

Chandresekhar Limit

We already saw that $\epsilon_F \sim m_e c^2$ for solar mass white dwarf - it is wrong to treat electron gas as nonrelativistic

Consider other extreme — Relativistic fermi gas, with $\epsilon \sim pc = \hbar c K$

still have $N/V \sim K_F^3$

$$U/V \sim \int^{K_F} dK\, K^2\, \hbar c K \sim (\hbar c)\, K_F^4$$

and $K_F = \epsilon_F/\hbar c \Rightarrow U/V \sim (\hbar c)^{-3} \epsilon_F^4$

(Compare Black body law: $U/V \sim (\hbar c)^{-3} \tau^4$)

So $U \sim \hbar c\, V\left(\frac{N}{V}\right)^{4/3} \sim \hbar c\, N^{4/3}/R$

For relativistic gas, kinetic energy, like potential energy, goes like $\frac{1}{R}$

Relativistic star will want to collapse, if heavy enough unstable for $\quad GM^2 \sim \hbar c \left(M/m_p\right)^{4/3}$

$$M^{2/3} \sim \left(\frac{\hbar c}{G}\right) m_p^{-4/3} \rightarrow M \simeq \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2}$$

and $\quad \frac{\hbar c}{G} = M_{planck}^2 \quad$ where $\quad M_{planck} \sim 10^{-5}\,g$

$$\boxed{M_{chandra} \sim M_{pl}^3/m_p^2}$$

$$\sim 10^{33}\,g \sim M_\odot$$

More careful analysis, considering nonuniform density of star, etc., finds maximum WD mass

$$M_{chandra} \sim 1.4\,M_\odot$$

– Really works

Neutron Star

A heavier star collapses -- to what?

Collapsed star can lower its energy by reducing $N_{electron}$, via

(Becomes favorable even though $m_n > m_p + m_e$)

$$p + e \rightarrow n + \nu_e$$

neutrinos escape – seen from SN1987a

Now can be supported by neutron degeneracy pressure

$$R \sim \frac{\hbar^2}{M_{neut}} \frac{1}{GM^2} \left(\frac{M}{m_n}\right)^{5/3}$$

smaller by $\frac{m_e}{m_n} \sim \frac{1}{1800} \rightarrow R \sim$ few km (!)

[Nuclear density!]

Grav potential energy released in explosion

$$\frac{GM^2}{R} = \left(\frac{GM/c^2}{R}\right) Mc^2 \sim Mc^2 \sim 10^{33}\,g\; 10^{21}\,cm^2 sec^{-2} \sim 10^{54}\,ergs \quad (!!)$$

(Because neutrons are relativistic at Chandra limit

(Mostly comes out as neutrinos)

Actually about 10^{53} ergs released in ~10 sec

Note -- Chandrasekhar argument again gives maximum mass $\sim M_{pl}^3/m_n^2$

But need corrections due to GR, nonideal gas (eqn of state), nonuniform density —

$$M_{\text{neutron star}} \simeq 3\,M_\odot$$

Coincidentally [Not really a coincidence -- at Chandra limit the K.E. of neutron comparable to mass; grav binding comparable to Mc^2]

$$R_{\text{Schwarzschild}} = \frac{2GM}{c^2} = 3\text{ km for } M \sim M_\odot$$

Neutron star close to being a <u>black hole</u>

Heavier star <u>will</u> collapse to BH