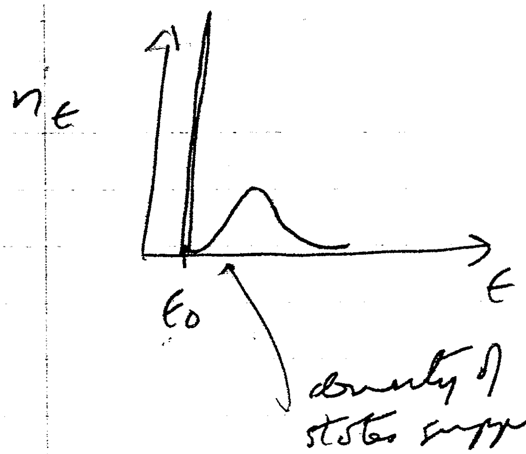
Bose-Einstein Condensation

The other quantum ideal gas, and quite different than Fermi gas

At $\tau = 0$, all particles are in ground state orbital. (Bosons, unlike fermions, are very gregarious.)
For $\tau \neq 0$, but small, they like to stay there, and most of them do.
Up to a critical temperature τ_E (Bose-Einstein condensation temp), a finite fraction of all particles remain in ground orbital.

This is quite different than black body distribution found for photons. The difference is that (we are assuming that) number N of particles is conserved. (Not true for photons in a cavity, but true for e.g. ^4He)



If we can't fit all N particles into excited states at some τ ($< \tau_E$), the rest go into ground orbital.

Why is that necessary?
$\Downarrow$ It is not.. really $|\mu| << \Delta E$ is true for BEC

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First suppose $\tau << \Delta E$ - the spacing between ground state (which we'll take to be at $\epsilon = 0$) and first excited. Then

$$f(\epsilon=0) = \frac{1}{e^{-\mu/\tau}-1} \sim N >> 1$$

so $$-\tau/\mu \cong N \Rightarrow \mu = -\tau/N$$

chemical potential will be negative and <u>very</u> close to zero. Activity

$$\lambda = e^{\mu/\tau} = e^{-1/N} \sim 1 - \frac{1}{N}$$

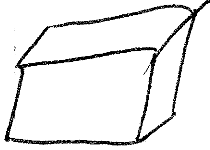
(<u>very</u> close to 1)

The key to Bose-Einstein condensation is that this chemical potential can satisfy

$$|\mu| << \Delta E$$

(and this is <u>still</u> true for $\tau >> \Delta E$, if τ is not <u>too</u> large) $\left[f(\varepsilon=\Delta\varepsilon) \sim \tau/\Delta\varepsilon << f(0) = \tau/|\mu|\right]$

As we increase volume, $N \propto$ volume (if concentration held fixed)



For particles in a box

$$\epsilon = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (n_x^2+n_y^2+n_z^2)$$

$$\Delta E = 3\frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2$$

So $\Delta E \sim \frac{1}{L^2}$ $\quad \mu \sim 1/L^3$

Example:

For ^4He $\quad m = 6.6\times10^{-24}\,g$, $L = 1\,cm$ $\Rightarrow$ $\Delta\epsilon = 2.5\times10^{-30}\,erg$

$N = 10^{22}$, $\tau = 1°K$ $\Rightarrow$ $-\mu = \tau/N = 1.4\times10^{-38}\,erg$

$$f(0) \sim -\tau/\mu \sim N$$

$$f(\Delta\epsilon) \sim \frac{1}{e^{\Delta\epsilon/\tau}-1} \sim \frac{\tau}{\Delta\epsilon} \qquad \text{(can ignore } \mu\text{)}$$

so
$$\frac{f(\Delta\epsilon)}{f(0)} \sim \frac{-\mu}{\Delta\epsilon} \sim \frac{1}{N}\left(\frac{\tau}{\Delta\epsilon}\right)$$

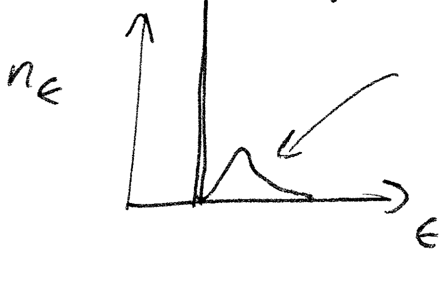
$\sim 10^{-8}$ for numbers above

contrast with <u>classical</u> formula

$$f(\epsilon) = \lambda e^{-\epsilon/\tau} \Rightarrow \frac{f(\Delta\epsilon)}{f(0)} \sim e^{-\Delta\epsilon/\tau} \sim 1$$

the quantum effects are <u>dramatic</u>

Now let's calculate more carefully; main thing is to correctly take into account the τ-dependence of μ



calculate $N_e(\tau)$ - the number of particles in excited states

Essentially the same as Planck, except for NR $\epsilon = p^2/2m$ instead of $\epsilon = pc$

we have $N = \sum_n f_n = N_0(\tau) + N_e(\tau)$

groundstate ↗   ↖ excited states

$$N_e(\tau) = \int_0^\infty d\epsilon\, \mathcal{D}(\epsilon)\, f(\epsilon,\tau)$$

$$\mathcal{D}(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{\frac{1}{2}}$$ for non-relativistic spinless bosons

$$f(\epsilon,\tau) = \frac{1}{e^{(\epsilon-\mu)/\tau}-1} = \frac{1}{\lambda^{-1}e^{\epsilon/\tau}-1}$$

so $$N_e(\tau) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon\,\frac{\epsilon^{1/2}}{\lambda^{-1}e^{\epsilon/\tau}-1}$$

Let $x = \epsilon/\tau$ $$= \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\tau^{3/2}\int dx\,\frac{x^{\frac{1}{2}}}{\lambda^{-1}e^{x}-1}$$

Note: $\propto \tau^{3/2}$ (cf τ^3 for relativistic bosons — photons)

$$I(\lambda) = \int_0^\infty dx\,\frac{x^{1/2}}{\lambda^{-1}e^{x}-1}$$

But $N_0(\tau) = \dfrac{1}{\lambda^{-1}-1}$ so $\lambda^{-1} = 1 + N_0^{-1}$

or $\lambda = (1+N_0^{-1})^{-1} \simeq 1 - \frac{1}{N_0}$

As long as $N_0 >> 1$, we have $\lambda \sim 1$, and make a small error by taking

$$I(\lambda) \sim I(1)$$

So $$I = \int_0^\infty dx \, \frac{x^{1/2}}{e^x - 1} = 1.306\sqrt{\pi}$$

and $$N_e(\tau) = \frac{V Z^{3/2}}{4\pi^{3/2}} 1.306 \left(\frac{m}{\hbar^2}\right)^{3/2} \tau^{3/2}$$

Recall that $$n_Q \equiv \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2}$$

$$\Rightarrow \boxed{\frac{N_e(\tau)}{V} \equiv n_e(\tau) = 2.61\, n_Q}$$

or $$\frac{n_e}{n} = 2.61 \frac{n_Q}{n}$$

We can use this formula as long as $N_0(\tau) >> 1$, or $n_e/n < 1$ (for a large volume)

For $$N_0(\tau) = V(n - n_e) = N\left(1 - \frac{n_e}{n}\right)$$

We have a finite fraction in the ground state as $V \to \infty$ until a critical Temp is reached where

$$n_e/n = 2.61 \frac{n_Q}{n} = 1$$

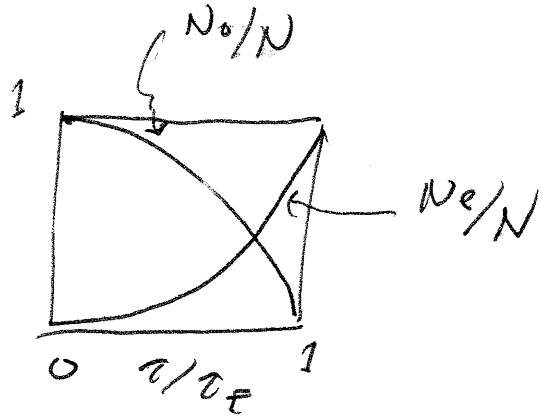
$$\left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2} = \frac{n}{2.61} \Rightarrow \boxed{\tau = \tau_E \equiv \frac{2\pi\hbar^2}{m}\left(\frac{n}{2.61}\right)^{2/3}}$$

For $\tau < \tau_E$, we have

$$\boxed{n_e/n = (\tau/\tau_E)^{3/2}}$$

For $\tau > \tau_E$, we'll have to have λ move away from 1, so $N_0 = O(1)$

Then we can have $n_e < 2.61\, n_Q$



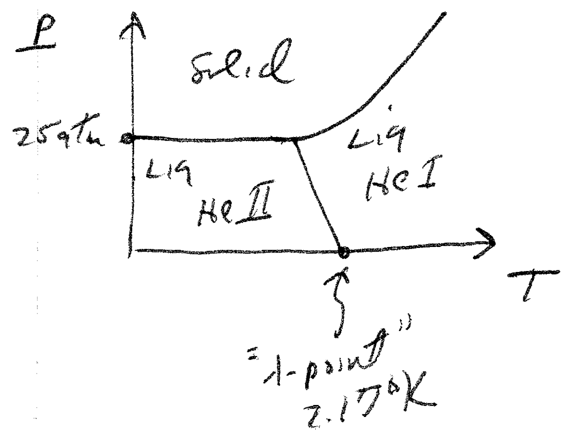
For ^4He, we take $\quad m = 6.6 \times 10^{-24}\, g$

$$n^{-1} = 27.6 \ cm^3 mol^{-1}$$

$$\Rightarrow \tau_E = 3.1° K$$

Liquid He II

^4He atoms are bosons, and interact relatively weakly. Because interactions are weak, and atoms are light, He does not solidify even at $\tau \to 0$, and atm pressure. Quantum-mechanical zero pt motion of the atoms "melt" the would be crystal lattice. It does liquefy, though.



Nothing special happens to Liq ^4He at 3.1°K

But something happens at 2.17°K (at low P)

(interactions change τ_E)

A new phase (He II) that has negligible viscosity! (flows without resistance)

Interpret as a two-fluid state:

superfluid = Bose condensate — pure groundstate

normal fluid = excited states

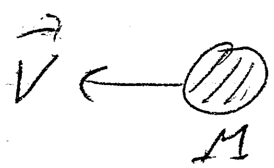
Note ^{3}He is also a liquid at $\tau = 0$
But behaves quite differently
^{3}He ~~bosons~~ atoms are fermions

Does show superfluidity, but only at $\tau \sim 10^{-3}\,{}^\circ K$
(pairing of atoms to form bosons that can condense).

Kinematics of Superfluidity

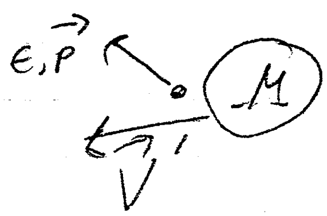
Superfluidity is not an automatic consequence of condensation. Actually need to consider interactions (e.g. hard core) to understand it.

Key is -- it is hard to pop a He atom out of the condensate.



consider heavy object moving through fluid. Normally, viscosity dissipates motion as heat.

This requires that "elementary excitations" can be excited by the moving object.



Kinematics

$$\tfrac{1}{2} M V^2 = \tfrac{1}{2} M V'^2 + \epsilon$$

$$M\vec V = M\vec V' + \vec p$$

$$M^2 V'^2 = M^2 \vec V^2 - 2M\vec V \cdot \vec p + p^2$$

$$= M^2 V^2 - \epsilon(2M)$$

or -- $\epsilon = \vec{V}\cdot\vec{p} - p^2/2M$

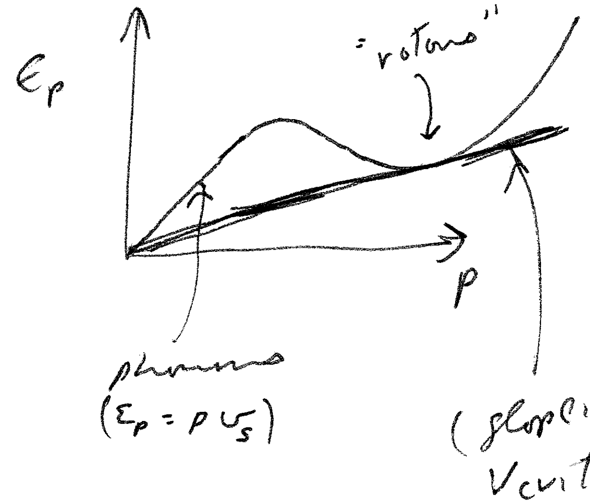
Suppose $M \to \infty \Rightarrow \boxed{\epsilon = \vec{V}\cdot\vec{p}}$

$\vec{V}\cdot\vec{p} < pV$, so if $\epsilon_p > pV$, no dissipation can occur. There is a critical velocity

$$V_{crit} = \text{minimum}\left(\epsilon_p/p\right)$$

For ordinary ideal gas, can excite one particle $\epsilon_p = p^2/2m$ - there is no min, and so $V_{crit} = 0$

But in Bose condensate, collective effects are important, for state is completely symmetrized



There are phonons $\epsilon_p = v_s\, p$ at low p, but no single particle excitations

The normal fluid does damp motion -- experiment regarding, say, flow through a tiny hole isolates superfluid component

Task of microscopic theory -- to explain this dispersion relation. [Ideal gas would not be superfluid: interactions are important