QUANTUM INFORMATION LECTURE, 27.02.2017

1. A stabilizer codes is a quantum code defined by a stabilizer group $\mathcal{S}$, which is an Abelian subgroup of the Pauli group on n qubits. We require that $-I \notin \mathcal{S}$. The codespace $\mathcal{C}$ is a subspace of the Hilbert space $(\mathbb{C}^2)^{\otimes n}$ spanned by $+1$-eigenvectors of stabilizers from $\mathcal{S}$. The number of logical qubits is $k = \log \dim \mathbb{C}$ and can be found as

$$k = n - \#(\text{independent generators of } \mathcal{S}). \tag{1}$$

2. The 2D toric code is an example of a stabilizer code. It is a topological code, i.e. it can be defined on a lattice and its stabilizer generators are geometrically local. In particular, the standard way of representing the toric code is on the square lattice with the X-vertex and Z-face terms. However, we can alternatively represent it on the (rotated) checker board: qubits are placed on vertices, and there are X- and Z-stabilizers associated with white and black faces, respectively.

3. The 2D color code can be defined on a lattice (with or without boundaries), which satisfies two conditions:

   - vertices are 3-valent,
   - faces are 3-colorable.

   Qubits are placed on vertices, and X- and Z-stabilizers are associated with every face of the lattice

4. Let V, E and F denote the set of vertices, edges and faces of the lattice. Using the properties of the lattice we can prove that stabilizers of the color code commute

$$\forall f_1, f_2 \in F : [X(f_1), Z(f_2)] = 0. \tag{2}$$

   This shows, that the stabilizer group of the color code (and thus the color code) is properly defined.

5. For every stabilizer code with the stabilizer group $\mathcal{S}$ we can define a stabilizer Hamiltonian

$$H = -\sum_{S \in \mathcal{S}} S, \tag{3}$$

where the sum is over a set of stabilizer generators. The ground space of the Hamiltonian H corresponds to the codespace. In particular, the stabilizer Hamiltonian of the 2D toric code is

$$H_{TC} = -\sum_{v\in V} X(v) - \sum_{f\in F} Z(f), \tag{4}$$

and the stabilizer Hamiltonian of the 2D color code is

$$H_{CC} = -\sum_{f\in F} X(f) - \sum_{f\in F} Z(f). \tag{5}$$

6. Violated stabilizers of the topological code can be thought of as point-like excitations, i.e. anyons. In particular, in the toric code we can only create excitations in pairs, unlike in the case of the color code, where it is possible to locally create excitations in triples. This is the reason why the color code is more challenging to analyze, e.g. from the perspective of error correction.

7. Relation between the toric code and the color code can be guessed by analyzing the structure of excitations. In particular, in the toric code we have two types of excitations, electric e and magnetic m, which correspond to violated X- and Z-stabilizers. The toric code excitations have the following fusion rules and non-trivial braiding statistics

$$e \times e = m \times m = 1, \tag{6}$$

$$(e, m) = -1. \tag{7}$$

In case of the color code, we have electric and magnetic excitations of three species, labelled by the color of faces of violated stabilizers stabilizers, e_R, e_G, e_B and m_R, m_G, m_B, with the following fusion rules and non-trivial braiding statistics

$$e_R \times e_G \times e_B = m_R \times m_G \times m_B = 1, \tag{8}$$

$$\forall i \in \{R, B, G\} : e_i \times e_i = m_i \times m_i = 1, \tag{9}$$

$$\forall i \neq j \in \{R, B, G\} : (e_i, m_j) = -1. \tag{10}$$

We can guess that there is a mapping between excitations of two copies $i = 1, 2$ of the toric code and the color code

$$e_1 \leftrightarrow e_R,\ \ m_1 \leftrightarrow m_B, \tag{11}$$

$$e_2 \leftrightarrow e_B, \; m_2 \leftrightarrow m_R, \tag{12}$$

$$\tag{13}$$

and thus there is a relation between these two models. Indeed, this intuition can be made rigorous: we can show that the color code is equivalent to two copies of the toric code. Here, we say that two codes to be equivalent if their stabilizer Hamiltonians are related by a local unitary transformation.

8. We can find the number of logical qubits encoded in the toric code or the color code to be

$$k_{TC} = 2g, \tag{14}$$

$$k_{CC} = 4g, \tag{15}$$

where g denotes the genus of the manifold on which we define the codes. In particular, for the toric code we have $n = |E|$ physical qubits, $|V| + |F|$ stabilizer generators and two relations between them

$$\prod_{v \in V} X(v) = \prod_{f \in F} Z(f) = I. \tag{16}$$

Using the Euler characteristic, $|V| - |E| + |F| = 2(1-g)$, we arrive at $k_{TC} = 2g$. Similarly, for the color code, there are $n = |V|$ physical qubits, $2|F|$ stabilizer generators and four relations between them

$$\prod_{f \in F_{RG}} X(f) = \prod_{f \in F_{RB}} X(f) = \prod_{f \in F_{RG}} Z(f) = \prod_{f \in F_{RG}} Z(f) = I. \tag{17}$$

Using the Euler characteristic, $|V| - |E| + |F| = 2(1-g)$, and the relation $|E| = \frac{3}{2}|V|$ we arrive at $k_{CC} = 4g$.

9. The color code can be defined on a lattice with boundaries. In particular, we consider the triangular color code which encodes $k = 1$ logical qubits. Logical operators can be thought of as strings connecting boundaries of all three colors. Note that a logical operator can split. We can think of implementing logical operators by creating some excitations in the bulk and dragging them to the boundaries where they can disappear. Splitting of a logical operator can be understood from the perspective of the fusion rule

$$e_R \times e_G \times e_B = m_R \times m_G \times m_B = 1. \tag{18}$$

10. It is desirable to have codes with transversal logical gates, since they do not spread errors in the uncontrollable way through the system. Here, a transversal unitary is defined to be a tensor product of single-spin unitaries. The color code has transversal logical X and Z operators, as well as logical H and S operators, where H is the Hadamard gate and $S = \text{diag}(1, i)$ is the phase gate. To prove that, we notice that $H^{\otimes n}$ and $S^{\otimes n}$ preserve the stabilizer group and they have the correct action (by conjugation) on the logical operators X and Z, thus implementing the correct logical operations.