S-Matrix and Verlinde Formula

Consider the Hopf link  $= \Psi_{ab}$

$\Psi_{ab} = \Psi_{ba}$

$\Psi^{*}_{ab} = \Psi_{\bar{a} b}$

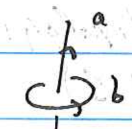
Modulus (up to normalization) captures amplitude for annihilation.

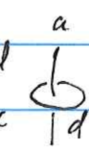
Phase ~ Monodromy

$S_{ab} = \frac{1}{\mathcal{D}} \Psi_{ab}$ is unitary if invertible ("modular"). — we'll show this below

 $= \#(a,b) \uparrow^{a}$ winding b does not change charge of a.

Close loop $\Psi_{ab} = \#(a,b)\, \Psi_{a1} \Rightarrow \#(a,b) = \frac{\Psi_{ab}}{\Psi_{a1}}$

 $= \frac{\Psi_{ab}}{\Psi_{a1}} \uparrow^{a} = \frac{S_{ab}}{S_{a1}} \uparrow^{a}$

we derived: (see page 2B)  $= \frac{S_{ab} S_{ac}}{S_{a1} S_{a1}} = \sum_d N_{bc}^{d} \frac{S_{ad}}{S_{a1}} \Big|^{a} = \sum_d N_{bc}^{d}$ 

Verlinde Relation $\Rightarrow \sum_d N_{bc}^{d} S_{da} = \frac{1}{S_{a1}} S_{ab} S_{ac}$ (and same for Ψ_{ab})

Denote $(\vec{S}_a)_d = S_{da}$ $\quad (N_b)_c^{\ d} = N_{bc}^{d}$

$$\Rightarrow (N_b) \vec{S}_a = \frac{S_{ab}}{S_{a1}} \vec{S}_a$$ "S diagonalizes fusion rules"

The basis $\vec{S}_a$ simultaneously diagonalizes all N_b

(they commute due to $(a \times b) \times c = a \times (b \times c)$)

Check: $(a\times b)\times c = N_{ab}^{d} N_{dc}^{e}\, e = (N_a N_c)_b^{e}\, e$

$a\times(b\times c) = N_{ad}^{e} N_{bc}^{d}\, e = (N_c N_a)_b^{e}\, e$

Verlinde generalizes $\quad N_b \vec{S}_1 = \dfrac{\psi_{1b}}{\psi_{11}} \vec{S}_1$, $\quad \dfrac{\psi_{1b}}{\psi_{11}} = d_b$ (largest eigenvalue)

(see next page) Follows from Verlinde formula.

N_a is real and normal $\Rightarrow$ $\perp$ eigenvectors.
But S might not be invertible.

Example: $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \det S = 0$ $\quad$ No nontrivial monodromy at all!

To exclude this, assume "modularity"
Every label except 1 has nontrivial monodromy with some other label (something topological distinguishes $a\neq 1$ from the label 1)

$N_b \vec{\psi}_a = \dfrac{\psi_{ab}}{\psi_{a1}} \vec{\psi}_a$ $\quad$ we assume $\forall a\neq 1\ \exists b$ such that $\dfrac{\psi_{ab}}{\psi_{a1}} \neq \dfrac{\psi_{1b}}{\psi_{11}}$

Therefore $\vec{\psi}_a$ and $\vec{\psi}_1$ are eigenvectors of N_b w/ distinct eigenvalues, hence $\perp$:

$\langle \vec{\psi}_1 | \vec{\psi}_a \rangle = 0$ for $a\neq 1$ $\quad \langle \vec{\psi}_1 | \vec{\psi}_1 \rangle = \mathcal{D}^2$

Consider Verlinde: $\psi_{ab}\psi_{ac} = \psi_{1a} \sum_d N_{bc}^{d} \psi_{da}$

And sum over a

$$(\psi^\dagger \psi)_{\bar{b}c} = \sum_a \psi^*_{\bar{b}a}\psi_{ac} = \sum_a \psi_{ab}\psi_{ac}$$

$$= \sum_d N_{bc}^{d} \sum_a \psi_{1a}\psi_{da} = \sum_a N_{bc}^{d}\, \delta_{1d} \langle \vec{\psi}_1 | \vec{\psi}_1 \rangle$$

$$= N_{bc}^{1}\, \mathcal{D}^2 = \delta_{\bar{b}c} \Rightarrow \quad S^\dagger S = I$$

Why N_b is normal

We want to show that its eigenvectors are either orthogonal or parallel.

Recall Verlinde: $\sum_d N_{bc}^{d} S_{da} = \frac{1}{S_{a1}} S_{ab} S_{ac}$

$$\Rightarrow \quad N_b \vec{S}_a = \frac{S_{ab}}{S_{a1}} \vec{S}_a$$

Therefore,

$$\frac{S_{ab}}{S_{a1}} (\vec{S}_c, \vec{S}_a) = (\vec{S}_c, N_b \vec{S}_a)$$

$$= \sum_{d,e} S^*_{ce} N_{be}^{d} S_{ad} = \sum_{d,e} S_{c\bar{e}} N_{b\bar{d}}^{\bar{e}} S_{ad}$$

But $\sum_{\bar{e}} N_{b\bar{d}}^{\bar{e}} S_{\bar{e}c} = \frac{1}{S_{c1}} S_{cb} S_{c\bar{d}}$

$$\Rightarrow \frac{S_{ab}}{S_{a1}} (\vec{S}_c, \vec{S}_a) = \sum_d \frac{S_{cb}}{S_{c1}} S_{c\bar{d}} S_{ad} = \frac{S_{cb}}{S_{c1}} (\vec{S}_c, \vec{S}_a)$$

Therefore, if $(\vec{S}_c, \vec{S}_a) \neq 0$ (not orthogonal), then

$$\frac{S_{ab}}{S_{a1}} = \frac{S_{cb}}{S_{c1}} \Rightarrow \vec{S}_a = \frac{S_{a1}}{S_{c1}} \vec{S}_c \quad \text{(parallel)}$$

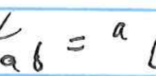
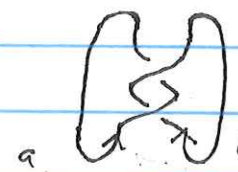
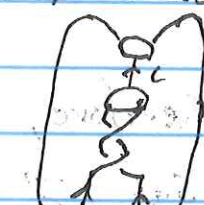
Therefore, in particular, if $\vec{S}_a$ and $\vec{S}_c$ are eigenvectors of N_b with distinct eigenvalues, then they cannot be parallel and must be orthogonal.

If S is invertible, then S determines N, since Verlinde relation implies

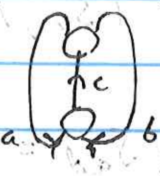
$$\sum_a \left(\sum_e N_{bc}^{e} S_{ea}\right) S^{\dagger}_{ad} = \sum_a \frac{1}{S_{a1}} S_{ab} S_{ac} S^{*}_{ad}$$

$$= \sum_e N_{bc}^{e}\, \delta_{ed} = N_{bc}^{d}$$

May we also say that N determines S? No, we need to know the topological spins as well:

$\Psi_{ab} =$  $=$  $= \sum_c$ 

To untwist, apply $(R^{c}_{ab})^{-2}$

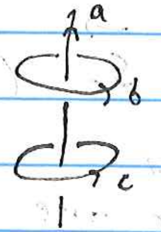
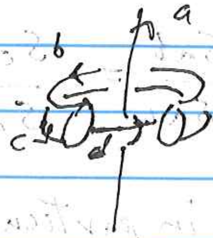
$= \sum_c \theta_c^{*} \theta_a \theta_b$  $= \sum_c \theta_c^{*}\theta_a\theta_b$ 

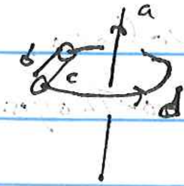
$$= \sum_c N_{ab}^{c}\, d_c\, \theta_a \theta_b \theta_c^{*}$$

$$\Rightarrow \quad S_{ab}\, \theta_a^{*} \theta_b^{*} = \sum_c N_{ab}^{c}\, d_c\, \theta_c^{*}$$

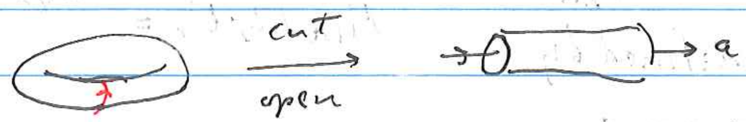
Verlinde Formula

For completeness, the derivation:

 $= \dfrac{S_{ab} S_{ac}}{S_{a1} S_{a1}}$  $= \sum_d$ 

$= \sum_d$  $= \sum_d N_{bc}^{d} \dfrac{S_{ad}}{S_{a1}}$ 

Another view of the S-matrix: Degeneracy of g.s. on the torus (a signature of topological order)



Cut the torus open. What label carried by puncture

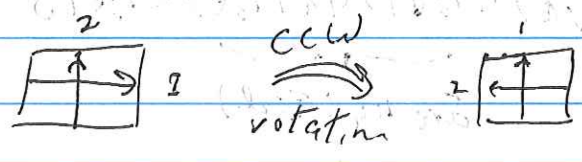
The # of states is # of labels.
(Modular ⇒ no further degeneracy.)

Alternative: cut the torus on the complementary cycle:

This defines another basis for the degenerate g. states.

We can define an "S-matrix" as the unitary transformation relating these bases.

We consider periodically identified parallelogram

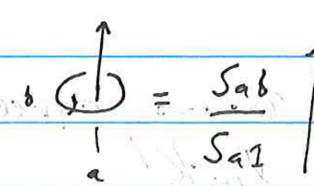


S: $C_1 \longmapsto C_2$
$C_2 \longmapsto -C_1$

(Hence S^2: $C_1 \to -C_1$, $C_2 \to -C_2$ Reverses both cycles

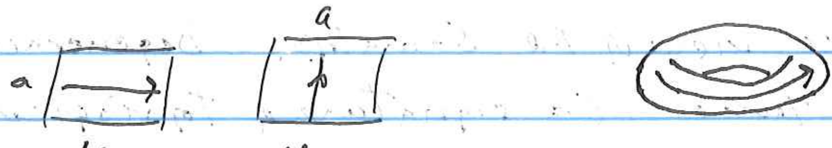
Then $S\,|1,a\rangle = |2,a\rangle \Rightarrow |1,a\rangle = S^{\dagger}|2,a\rangle$
$S\,|2,a\rangle = |1,\bar{a}\rangle \Rightarrow |1,a\rangle = S\,|2,\bar{a}\rangle$

Is this the same as unitary defined by



? We can verify that it is!

Define operators which preserve ground space



M_{1a} $\quad$ M_{2a}

operators defined by winding anyons around torus

$$M_{1a}|2;b\rangle = \frac{S_{\bar{a}b}}{S_{1b}}|2,b\rangle$$

$$M_{2a}|1,b\rangle = \frac{S_{ab}}{S_{1b}}|1,b\rangle$$

Lets define eigenvalues $M_{1a}|2;b\rangle = m_1(\bar{a};b)|2,b\rangle$

$M_{2a}|1,b\rangle = m_2(a;b)|1,b\rangle$

We want to relate to S matrix that interchanges the cycles.

use $N_{ab}^{c} = \langle 1;c|M_{1a}|1;b\rangle$

$$= \langle 2;c|S M_{1a} S^{+}|2;b\rangle$$

$$= \sum_{d,e} \langle 2;c|S|2,d\rangle \underbrace{\langle 2;d|M_{1a}|2;e\rangle}_{\delta_{de}\, m_1(\bar{a};d)} \langle 2;e|S^{+}|2;b\rangle$$

$$= \sum_{d} S_{cd}\, m_1(\bar{a};d)\, S^{+}_{db}$$

Now suppose $c=1 \Rightarrow N_{ab}^{1} = \delta_{\bar{a}b} = \sum_{d} S_{1d}\, m_1(\bar{a};d)\, S^{+}_{db}$

Multiply by S_{bf} and $\sum_{b}$

$$\Rightarrow S_{\bar{a}f} = S_{1f}\, m_1(\bar{a};f) \Rightarrow m_1(\bar{a};f) = \frac{S_{\bar{a}f}}{S_{1f}}$$

It works: Modular S on torus agrees with S defined by Hopf link!

Ising Anyons and Majorana Modes

Ising anyons: There are 3 labels: $1, \psi, \sigma$

where ψ is a fermion and

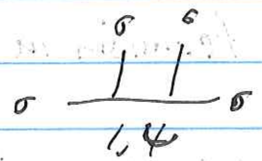
$\psi \times \sigma = \sigma$

$\sigma \times \sigma = 1 + \psi$

$\psi \times \psi = 1$

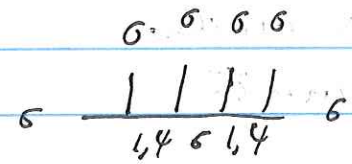
$\Rightarrow d_\sigma = \sqrt{2}$

Standard basis for 4 σ's is (total charge 1)



Encodes a qubit

Add more qubits



So 2 σ's $\simeq$ 1 qubit

For n qubits $N^1_{n\sigma} = 2^{(n-2)/2}$ i.e. $\frac{n-2}{2}$ qubits

Kitaev solved the ~~hexagon~~ pentagon + hexagon equation: 8 solutions

$$F^\sigma_{\sigma\sigma\sigma} = \varkappa_\sigma \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad R_{\sigma\sigma} = \varkappa_\sigma e^{-i\pi\nu/8} \begin{pmatrix} 1 & 0 \\ 0 & i^\nu \end{pmatrix}$$

± 1 $\quad$ ν is odd $\quad |\nu| = 1, 7, \quad \varkappa_\sigma = 1$

$|\nu| = 3, 5, \quad \varkappa_\sigma = -1$

The σ has topological spin $e^{i\theta} = e^{i\pi\nu/8} = e^{2\pi i\nu/16}$ $\quad \nu = \text{odd}$

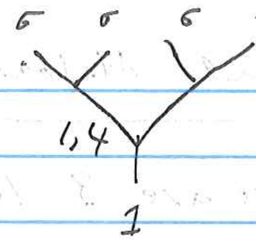
All the solutions have the form

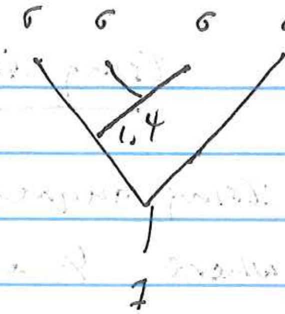
$$F = \pm\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{Hadamard}$$

$$R = \text{phase} \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix} \sim \exp\left(-\frac{i\pi}{4} Z\right) \quad \text{90° rotation about } Z \text{ axis}$$

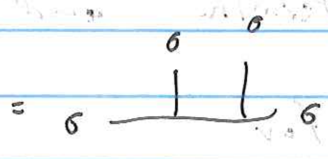
F and R generate single qubit Clifford

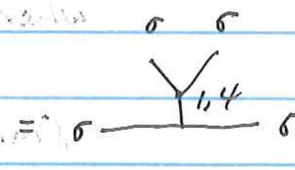
z-basis





x-basis





Majorana modes: Divide a fermion in half

Fermion modes anticommute

$$0 = a_i a_j + a_j a_i = a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger$$

$$a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}$$

For a single mode $a^2 = 0 = a^{\dagger 2}$

$$a a^\dagger = 1 - a^\dagger a$$

Define $|0\rangle$ by $a|0\rangle = 0$

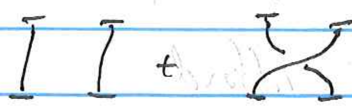
$$a^\dagger |0\rangle = |1\rangle$$

$a^\dagger a|0\rangle = 0$ $\quad a^\dagger a|1\rangle = a^\dagger a a^\dagger |0\rangle = a^\dagger(1 - a^\dagger a)|0\rangle = |1\rangle$

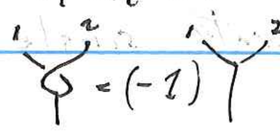
⇒ occupation # $a^\dagger a = 0, 1$

A fermionic mode encodes 1 qubit.
Except -- there is $(-1)^F$ superselection rule
observables have even fermionic parity — that is,
$(-1)^F$ is locally conserved.

For two modes the minus sign $a_1 a_2 = -a_2 a_1$
keeps track of the fermionic exchange phase



$a_1^\dagger a_2^\dagger |0\rangle = -a_2^\dagger a_1^\dagger |0\rangle$



Majorana mode: Split a into Hermitian and anti-Hermitian part

$$c_1 = a_1 + a_1^\dagger \qquad \Rightarrow \quad c_1^2 = a_1 a_1^\dagger + a_1^\dagger a_1 = 1$$
$$c_2 = -i(a_1 - a_1^\dagger) \qquad\quad c_2^2 = -(-a_1 a_1^\dagger - a_1^\dagger a_1) = 1$$

$$c_1 c_2 = -i(-a_1 a_1^\dagger + a_1^\dagger a_1) = -i(2a_1^\dagger a_1 - 1)$$
$$c_2 c_1 = -i(-a_1^\dagger a_1 + a_1 a_1^\dagger) = -c_1 c_2$$

$$\Rightarrow \quad -i(c_1 c_2) = (1 - 2a_1^\dagger a_1) = \begin{matrix} +1 & \text{for } a_1^\dagger a_1 = 0 \\ -1 & \text{for } a_1^\dagger a_1 = 1 \end{matrix}$$

$$= (-1)^{F_{12}} \quad \text{fermionic parity}$$

Note that $(-ic_1c_2)^2 = -c_1c_2c_1c_2 = c_1^2 c_2^2 = +1$

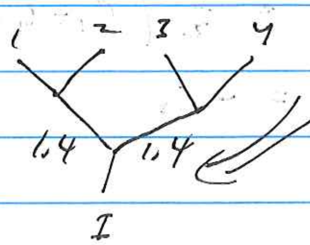
The Majoranas
- Anticommute
- Square to one
- Are Hermitian

Normally, Majorana modes are paired, but under the right physical conditions they can be unpaired (we'll see an example soon).

Then a single qubit can be stored in a pair of modes. Except it is not really a qubit, because the two states are in distinct $(-1)^F$ superselection sectors

But we can encode a qubit in 4 Majorana modes with $(-1)^F = 1$

where $(-1)^F$
$= (-ic_1c_2)(-ic_3c_4)$
$= -c_1c_2c_3c_4$



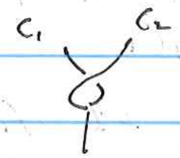
this is fusion rule

$\sigma \times \sigma = 1 + \psi$

1, 4 have $(-1)^F = \pm 1$; carry distinct values of a locally conserved charge

We can infer F and R matrices, which should be one of the solns to Pentagon + Hexagon eqn we found.

First consider R matrix



$$c_1 \mapsto s c_2 = c_1'$$
$$c_2 \mapsto t c_1 = c_2'$$

we must have $c_1'^2 = c_2'^2 \Rightarrow s^2 = t^2 = 1$

And $(-1)^F = -i c_1' c_2' = -i(st c_2 c_1) = (-st)(-i c_1 c_2)$ must be conserved, hence $st = -1 \Rightarrow$

$$s = \pm 1 \qquad t = \mp 1$$

Two solns $\quad \begin{matrix} c_1 \mapsto c_2 \\ c_2 \mapsto -c_1 \end{matrix} \qquad \begin{matrix} c_1 \mapsto -c_2 \\ c_2 \mapsto c_1 \end{matrix}$ These are inverses of one another

So one is R and other is R^{-1}. Which is which is a convention

We may write $\quad R_{\sigma\sigma} = \frac{1}{\sqrt{2}}(I + c_1 c_2)$

$\qquad R_{\sigma\sigma}^{-1} = \frac{1}{\sqrt{2}}(I - c_1 c_2)$

up to a phase! Since $(c_1 c_2)^2 = -1$, we may also write $R = \exp\left(\frac{\pi}{4} c_1 c_2\right)$ (up to phase)

We can check:

$$R c_1 R^{-1} = \tfrac{1}{2}(1 + c_1 c_2) c_1 (1 - c_1 c_2)$$
$$= \tfrac{1}{2}\left(c_1 + c_1 c_2 c_1 - c_1^2 c_2 - c_1 c_2 c_1 c_1 c_2\right)$$
$$= -c_2$$

$$R c_2 R^{-1} = \tfrac{1}{2}(1 + c_1 c_2)\, c_2\, (1 - c_1 c_2)$$
$$= \tfrac{1}{2}\left(c_2 + c_1 c_2^2 - c_2 c_1 c_2 - c_1 c_2 c_2 c_1 c_2\right)$$
$$= c_1$$

Also $R^2_{\sigma\sigma} = \tfrac{1}{2}(1 + c_1 c_2)(1 + c_1 c_2)$
$$= \tfrac{1}{2}(1 + 2c_1 c_2 + c_1 c_2 c_1 c_2) = c_1 c_2 = \begin{cases} i & 1 \\ -i & 4 \end{cases}$$

This means topological spin of σ is $e^{i\theta_\sigma} = -i$?
No, there is an overall phase of R that fixes the topological spin!

What about the F-matrix. To relate the two bases, it is convenient to use the Jordan-Wigner transformation, which relates $2n$ Majorana modes to n-qubit Pauli operators. For $n=2$, the transformation is

$$c_1 = XI \qquad c_2 = YI \qquad c_3 = ZX \qquad c_4 = ZY$$

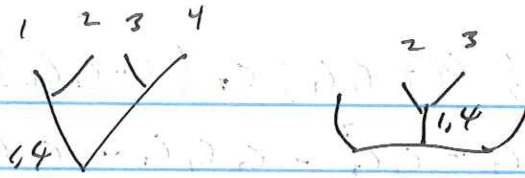
These (1) Are Hermitian, (2) Square to I, (3) are mutually anticommuting.

Fermionic parity $\quad (-1)^F = -c_1 c_2 c_3 c_4 = ZZ$

E.g. for $(-1)^F = 1$, the qubit is span$\{|00\rangle, |11\rangle\}$, the 2-qubit repetition code

$$-i c_1 c_2 = ZI = \bar{Z} \qquad -i c_2 c_3 = XX = \bar{X}$$

These are the logical Pauli operators of the code. Furthermore, any physical operator has $(-1)^F = 1$, and so preserves the code space



The two bases related by $\bar{Z} \leftrightarrow \bar{X}$ a logical Hadamard

$$F = (\text{phase}) \times \bar{H}$$

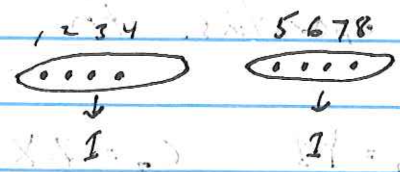
applying R to 1 and 2:

$$R = \exp\left(\frac{\pi}{4} c_1 c_2\right) = \exp\left(i \frac{\pi}{4} \bar{Z}\right)$$

applying R to 2 and 3 $\quad R = \exp\left(\frac{\pi}{4} c_2 c_3\right) = \exp\left(i \frac{\pi}{4} \bar{X}\right)$

These are $\frac{\pi}{2}$ rotations about z and x axes, which generate the 1-qubit Clifford group.

How do we do an entangling gate on a pair of topological qubits? Consider the initial state $|\bar{0}, \bar{0}\rangle$ with $\bar{Z}_1 = \bar{Z}_2 = 1$



we have 2 clusters of 4 qubits each, where $(-1)^F = 1$ in each cluster

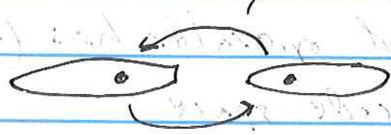
Qubits are encoded in the space with

$$-c_1 c_2 c_3 c_4 = 1 = -c_5 c_6 c_7 c_8$$

The initial state is

$$\bar{Z}_1 = 1 = -i c_1 c_2 = -i c_3 c_4$$
$$\bar{Z}_2 = 1 = -i c_5 c_6 = -i c_7 c_8$$

Can we reach an entangled state with an appropriate braid? Exchanging particles within any cluster of 4 preserves the product structure



So consider exchanging particles in different clusters

The initial state is stabilized by

$$1 = -ic_1c_2 = -ic_3c_4 = -ic_5c_6 = -ic_7c_8$$

The braid will transform

$$-ic_ic_j \longmapsto \mp i c_k c_\ell$$

If k,ℓ belong to the same qubit, then state is an eigenstate of one of $\bar{X}, \bar{Y}, \bar{Z}$

Hence, if our encoding is preserved ($(-1)^F = 1$ for each cluster) the state is a product state.

If k, ℓ are in different clusters, then

$-ic_kc_\ell$ anticommutes with $-c_1c_2c_3c_4$
$-c_5c_6c_7c_8$

This means the state is <u>not</u> an eigenstate of $(-1)^F$ for each cluster. It is outside our qubit encoding. The trouble is -- our braiding operations cannot realize an encoded entangling gate.

To do a CNOT --- Recall it suffices to do

(1) Pauli ops Z and X
(2) Prep of $|0\rangle$
(3) Destructive meas. of X
(4) Nondestructive meas. of XX and ZZ

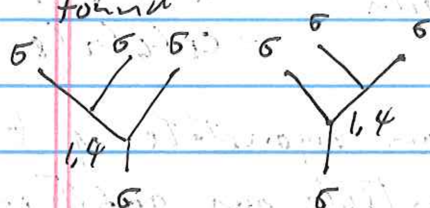
We know how to do (1), (2), (3); what we need is (4).

E.g. ZZ measurement means finding the total charge of Maj modes 3456

$\bar{Z}_1 \bar{Z}_2 = -c_3c_4c_5c_6$ $\qquad$ $\bar{X}_1 \bar{X}_2 = -c_2c_4c_5c_7$

Even if we can do these nondestructive measurements robustly, we would also need a non-Clifford gate (e.g. T) which would not be topologically protected.

How do we braid Majoranas? In fact it suffices to be able to measure fermionic parity (I or ψ) for a pair of modes. We use the concept of a "forced measurement." The measurement can be repeated until the desired outcome is found.



Measuring $(-1)^F$ for a pair of modes is like measuring Z of a qubit when performed on modes 1 and 2, while measurement of parity of modes 2 and 3 is an X measurement. If we want to prepare $|0\rangle$ we measure Z, if outcome is $|1\rangle$ we measure X and then repeat Z measurement. In each round we have success prob. $\frac{1}{2}$.

To braid modes 1 and 2, introduce ancilla modes A and B in the state with $(-1)^F = 1$. Then we perform 3 forced measurements in succession, each time projecting a pair of modes onto the $(-1)^F = 1$ state. If Π_{AB} denotes the projection onto the state with $-ic_Ac_B = 1$, we have $\Pi_{AB} = \frac{1}{2}(I - ic_Ac_B)$.

Note that

$$\Pi_{AB}\,\Pi_{B1}\,\Pi_{B2}\,\Pi_{AB} = \Pi_{AB}\,\tfrac{1}{4}(I - i c_B c_1)(I - i c_B c_2)\,\Pi_{AB}$$

$$= \Pi_{AB}\,\tfrac{1}{4}\left(I - c_B c_1 c_B c_2 - i c_B c_1 - i c_B c_2\right)\Pi_{AB}$$

But $\Pi_{AB}(c_B c_1)\Pi_{AB} = 0 = \Pi_{AB}(c_B c_2)\Pi_{AB}$
because $c_B c_1$ and $c_A c_B$ anticommute

$$= \Pi_{AB}\,\tfrac{1}{4}\left(I + c_1 c_2\right)\Pi_{AB} = \frac{1}{2^{3/2}}\,R_{12}\,\Pi_{AB}$$

The factor of $\frac{1}{2^{3/2}}$ reflects that each of 3 measurements succeeds with amplitude $\frac{1}{\sqrt{2}}$ (probability $\frac{1}{2}$)

It is not really necessary to "force" the measurement of B1 parity and B2 parity; if these yield outcomes $\alpha_1, \alpha_2 \in \{\pm\}$, then the measurement protocol yields

$$\frac{1}{\sqrt{2}}\left(I + \alpha_1 \alpha_2\, c_1 c_2\right);$$

that is, either R_{12} or R_{12}^{-1} depending on whether the parity measurements agree or disagree. It really is necessary to force the final AB parity measurement, though. If we get the wrong outcome we have flipped the fermionic parity of the mode pair 1,2.

Measuring the parity for a set of 4 modes allows us to measure ZZ for a pair of Majorana qubits — which suffices for realizing an entangling

Two-qubit Clifford gate. We can show this directly by devising a protocol which implements

$$W_{1234} = \exp\left(i\frac{\pi}{4} c_1 c_2 c_3 c_4\right) = \frac{1}{\sqrt{2}}\left(I + i\, c_1 c_2 c_3 c_4\right)$$

Applied to the 4 modes from 2 qubits as shown, this is $e^{-i\pi/4(Z_1 Z_2)}$ applied to 2 qubits, or $e^{i\pi/4}\mathrm{diag}(1, i, i, 1)$.

This is an entangling Clifford gate, related by 1-qubit Clifford gates to $\Lambda(Z)$, as

$$\Lambda(Z) = \exp\left(-i\frac{\pi}{4}(I - Z_1)(I - Z_2)\right)$$
$$= e^{i\pi/4\, Z_1} e^{i\frac{\pi}{4} Z_2} e^{-i\pi/4\, Z_1 Z_2}$$

Again we introduce an ancilla pair of anyons, initially w/ $(-1)^F = I$, and perform a sequence of 4 forced measurements:

$$\Pi_{AB}\, \Pi_{A1}\, \Pi_{1234}\, \Pi_{B1}\, \Pi_{AB}$$
$$= \Pi_{AB}\, \frac{1}{8}\left(I - i c_A c_1\right)\left(I - c_1 c_2 c_3 c_4\right)\left(I - i c_B c_1\right)\Pi_{AB}$$
$$= \Pi_{AB}\, \frac{1}{8}\Big(I - i c_A c_1 - i c_B c_1 - c_A c_1 c_B c_1$$
$$- c_1 c_2 c_3 c_4 + i c_A c_1 c_1 c_2 c_3 c_4 + i c_1 c_2 c_3 c_4 c_B c_1$$
$$+ c_A c_1 c_1 c_2 c_3 c_4 c_B c_1\Big)\Pi_{AB}$$

Setting $c_A c_B = i$, and dropping terms that vanish inside $\Pi_{AB}(\;)\Pi_{AB}$

$$= \Pi_{AB}\frac{1}{8}\left(I + i - c_1 c_2 c_3 c_4 + i c_1 c_2 c_3 c_4\right)\Pi_{AB} = \frac{1}{4}\Pi_{AB}\left(\frac{1+i}{\sqrt{2}}\right)\frac{1 + i c_1 c_2 c_3 c_4}{\sqrt{2}}\Pi_{AB}$$
$$= \frac{1}{4} e^{i\pi/4} W_{1234}\, \Pi_{AB}$$

If the measurement outcomes are

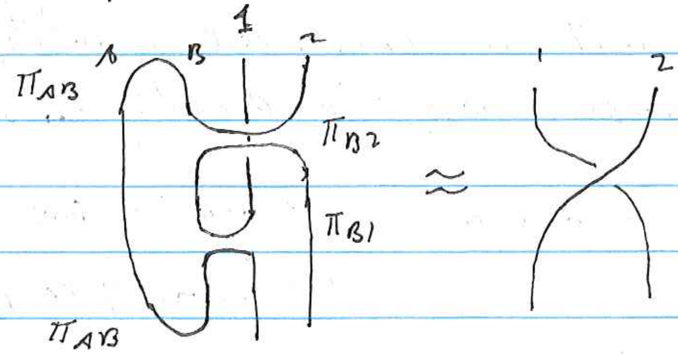
$\Pi_{A1} = a \quad \Pi_{1234} = b \quad \Pi_{B1} = c$ where $a, b, c \in \{\pm\}$,

the operation becomes

$$\frac{1}{4}\left(\frac{1+aci}{\sqrt{2}}\right)\left(\frac{I + b\, i c_1 c_2 c_3 c_4}{\sqrt{2}}\right) \Pi_{AB}$$

Hence there is no need to force the 1st and third measurements; the outcomes only affect the overall phase. The outcome of the four-mode measurement determines whether the operation is W_{1234} or W_{1234}^{-1}. These two differ by $\mathrm{diag}(1 -1 -1 1)$, which is just the Pauli operator $Z_1 Z_2$. So even the 4-mode measurement need not be forced if we are willing to update the Pauli frame. The final forced AB measurement is necessary, however.

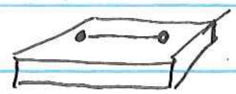
How "braiding by measurement" works is illuminated by this diagram:



However, it is hard to tell from the diagram whether R or R^{-1} is realized — whether strand 1 passes above or below 2

Next, we want to understand how unpaired Majoranas can be realized in a physical device...

Unpaired Majoranas in Quantum Wires



The basic device is a "quantum wire" sitting atop a superconducting substrate

Under suitable conditions, unpaired Majoranas appear at the ends of the wire. That is, there are two nearly degenerate states of the wire, with $(-1)^F = \pm 1$, which are nearly indistinguishable. With two such wires we can encode a qubit:

Both with overall $(-1)^F = 1$.

To realize unpaired modes, we need two physical ingredients.

① "Spin-orbit coupling." This allows us to ignore the electron spin; we may consider the particles in the wire to be spinless fermions

② Superconductivity. This means that fermion number is not a conserved quantity (though $(-1)^F$ must of course always be locally conserved).

$$\overset{1}{\circ}\ \overset{2}{\circ}\ \overset{3}{\circ}\ \circ \text{ --- } \circ\ \circ\ \circ$$

We model the wire as a 1D array of coupled fermionic modes.

$$H = H_{hop} + H_{pot} + H_{pair}$$

$$= \sum_j \left[-w\left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j\right) - \mu a_j^\dagger a_j + \Delta a_j a_{j+1} + \Delta a_{j+1}^\dagger a_j^\dagger \right]$$

We reexpress a_j and a_j^+ in terms of Majorana modes

$$a_j = \frac{1}{2}(c_{2j-1} + i c_{2j}) \qquad a_j^+ = \frac{1}{2}(c_{2j-1} - i c_{2j})$$

Then "chemical potential term" is e.g.

$$n_1 = a_1^+ a_1 = \frac{1}{2}(1 + i c_1 c_2) = \begin{cases} 0 & \text{for } -ic_1c_2 = 1 \\ 1 & \text{for } -ic_1c_2 = -1 \end{cases}$$

That is, up to an additive constant.

$$H_{pot} = -\frac{i}{2}\mu(c_1c_2 + c_3c_4 + \cdots) + \cdots$$

For the hopping term we have

$$a_1^+a_2 + a_2^+a_1 = \frac{1}{4}\left[(c_1 - ic_2)(c_3 + ic_4) + (c_3 - ic_4)(c_1 + ic_2)\right]$$

$$= \frac{i}{2}(c_1c_4 - c_2c_3)$$

$$\Rightarrow \quad H_{hop} = \frac{i}{2}\omega(c_1c_4 - c_2c_3 + c_5c_8 - c_6c_7 + \cdots)$$

For the pairing term we have

$$a_1a_2 + a_1^+a_2^+ = \frac{1}{4}\left[(c_1 + ic_2)(c_3 + ic_4) + (c_3 - ic_4)(c_1 - ic_2)\right]$$

$$= \frac{i}{2}\left[c_1c_4 + c_2c_3\right]$$

$$\Rightarrow \quad H_{pair} = \frac{i}{2}\Delta(c_1c_4 + c_2c_3 + \cdots)$$

The potential term pairs up modes in same fermion mode

The hopping and pairing terms pair up Majoranas from different fermionic modes:

Adding the terms together we obtain

$$H = \frac{i}{2}\Big[-\mu(c_1c_2 + c_3c_4) + (w+\Delta)(c_1c_4) + (w-\Delta)\,c_2c_3\Big]$$

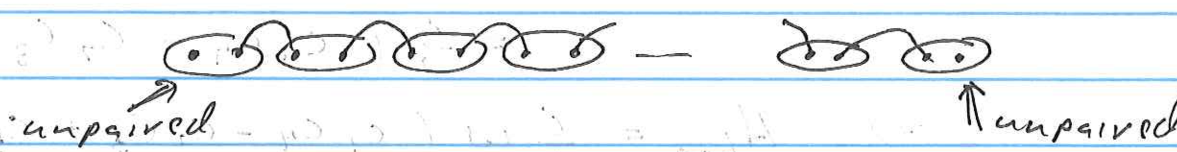
Consider two limiting cases:

① $w=\Delta=0$, $\mu<0$

Each fermionic mode has $n_j=0$ $\;(-ic_{2j-1}c_{2j}=1)$ in the ground state.

② $\mu=0$, $w=-\Delta>0$

$$\Rightarrow \quad H = \frac{i}{2}(2w)\left(c_2c_3 + c_4c_5 + c_6c_7 + \cdots\right)$$



Now it is pairs of Majoranas from neighboring site that have definite $(-1)^F$ in the g.s. C_1 and C_{2n} do not appear in the Hamiltonian at all. These are the unpaired Majoranas. There are 2 ground states with fermionic parity $-ic_1c_{2n}=\pm 1$, which are exactly degenerate, and locally indistinguishable. (Both also have definite value of total $(-1)^F = \prod_j(-ic_{2j-1}c_{2j})$.)

How robust is the degeneracy when we perturb the Hamiltonian from this special point?

It is helpful to recall the Jordan Wigner transformation, which maps $2n$ Majorana modes to n qubits:

$$c_1 = XI\cdots \qquad c_3 = ZXI\cdots \qquad c_5 = ZZXI\cdots$$
$$c_2 = YI\cdots \qquad c_4 = ZYI\cdots \qquad c_6 = ZZYI\cdots$$

$$\Rightarrow \quad -ic_1c_2 = ZI\cdots \qquad -ic_3c_4 = IZI\cdots$$
$$-ic_2c_3 = XXI\cdots \qquad -ic_4c_5 = IXXI\cdots$$
$$-ic_1c_4 = -YY \qquad -ic_3c_6 = -IYYI\cdots$$

In terms of Pauli operators the Hamiltonian becomes

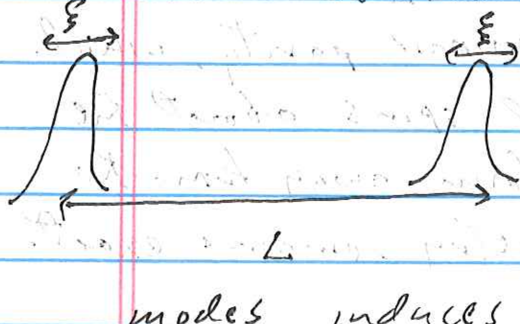
$$H = \frac{\mu}{2}(Z_1+Z_2+Z_3+\cdots) + \frac{(\Delta-w)}{2}(X_1X_2 + X_2X_3 + \cdots)$$
$$+ \frac{\Delta+w}{2}(Y_1Y_2 + Y_2Y_3 + \cdots)$$

In the limiting case where $\mu = \Delta + w = 0$, this is just a classical Ising model with XX nearest-neighbor couplings. The two exactly degenerate states are the magnetized states with all spins up or down along the x-axis

For general values of the parameters, the model has a symmetry: $Z_1Z_2\cdots Z_n$ commutes with H. This is just the fermionic parity, which simultaneously flips all spins about the z-axis. When we deform away from the special point, this symmetry remains exact.

When we turn on the Z_j term, it creates a domain wall at neighboring links, or moves a domain wall one step along the chain. Similarly, $Y_j Y_{j+1}$ creates a pair of walls 2 sites apart, or moves a wall 2 sites along the chain. If these perturbations are weak, then the degeneracy is split by only a small amount. The splitting arises from a tunneling process, in which a virtual domain wall sweeps across the chain, and scales like $\exp(-(\text{const})L)$, where L is the length of the chain.

We discussed a similar phenomenon when considering the degeneracy in a symmetry-protected topological phase. Here, too, a perturbation that breaks the Z_2 symmetry can lift the degeneracy, for example the perturbation X_j can distinguish the spin-up and spin-down magnetized phases. But $X_j = c_{2j-1}$ has $(-1)^F = -I$; it is not a physically realizable operator. The symmetry $(-1)^F$ that protects the degeneracy cannot be broken!



Another way to think about this: the perturbation broaden the unpaired Majorana mode, which decays as $e^{-\text{distance}/\xi}$, and the overlap of two modes induces a coupling $H_{eff} = t(-i c_1 c_n)$ where $t \sim e^{-L/\xi}$.