"One-way" quantum computer (measurement based universal computation with cluster states)

Note Title 2/17/2017

We can do universal quantum computation by performing single-qubit measurements, if the measurements are performed on a suitably prepared entangled resource state.

An example is a cluster state, or more generally a graph state. Let's first consider the cluster state associated with a one-dimensional lattice of qubits, and see how to execute a universal set of single-qubit gates. Then we'll extend the discussion to a two-dimensional lattice, and see that we can do a CNOT gate as well, completing a universal gate set.

A cluster state is a stabilizer state, a simultaneous eigenstate with eigenvalue 1 of a set of commuting Pauli operators. In the one dimensional lattice, there is a stabilizer generator associated with the ith qubit, namely ZXZ acting on qubits i-1, i, i+1, unless the qubit is at the end of the chain. For the first qubit the stabilizer generator is XZ acting on qubits 1, 2, and for the nth (last) qubit is is ZX acting on qubits n-1 and n.

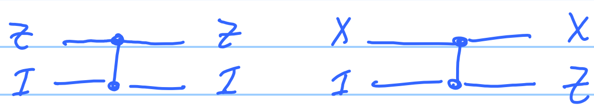
These stabilizers are mutually commuting. The generators i and i+1 collide on two qubits, and ZX commutes with ZX. The generators i and i+2 collide on a single qubit, where each applies Z. Since there are n qubits and n independent stabilizer generators, there is a unique state satisfying these conditions.

In order to have a single encoded qubit, let's eliminate the first stabilizer generator on the left edge of the lattice. Now we have a k=1 code. We can choose its encoded Pauli operators (which commute with the stabilizer and anticommute with one another) to be:

$$\bar{Z} = Z\, I\, I \cdots I$$
$$\bar{X} = X\, Z\, I \cdots I$$

This code has distance d = 1, so its error correcting power is not impressive, but we would like to consider how the encoded information propagates through the lattice as we measure the qubits one at a time, starting at the left edge and working our way to the right.

But first let's notice that it is very easy to prepare this 1D cluster state. Recall the action by conjugation of the controlled-Z gate on Pauli operators.



Suppose that the initial state is a tensor product of n X-eigenstates |+>, and then controlled-Z is applied to each pair of neighboring qubits. The controlled-Z gates acting on different pairs of qubits are mutually commuting, so all can be applied in parallel in a single time step.

The controlled-Z gates transform the stabilizer IXI acting on three successive qubits to the stabilizer ZXZ of the cluster state. We can apply the same construction to any graph: Starting with |+> at each vertex of the graph, we apply controlled-Z to each pair of qubits connected by an edge. The corresponding state is called a graph state. The term cluster state is used when the graph is a regular lattice, like the 1D chain, or a 2D square lattice.

What happens if we measure one of the qubits in the Z basis? If we measure the first qubit, we are just measuring the logical qubit in the Z basis. If we measure any other qubit, the stabilizer generator ZXZ is replaced by a new stabilizer generator IZI. This has the effect of breaking the entangled cluster state into the product of two cluster states, lying to the left and to the right of the measured qubit.

The effect of measuring in the X basis is more interesting: Instead of splitting the cluster state in two, measuring X on a string of qubits glues together the chain to the left of the measured qubits with the chain on the right, yielding a single cluster state. For the chain with a logical qubit at the left end, suppose we measure the first qubit in the X basis. This does not measure the logical qubit; rather the logical qubit of the length-n chain is tranformed to the logical qubit on a chain of length n-1, and furthermore a nontrivial rotation is applied to the logical qubit. Recall that, before the measurement, we can multiply by a stabilizer element to obtain an equivalent form of the logical Z. Then if the measurement of X yields outcome (-1)^a:

$$\bar{Z} = ZI\cdots I \equiv IXZI\cdots I \Rightarrow XZI\cdots I = \bar{X}$$

$$\bar{X} = XZI\cdots I \Rightarrow (-1)^a ZI\cdots I = (-1)^a \bar{Z}$$

We can now act on the second qubit with X^a (that is, do nothing if the measurement outcome is +1 and apply X if the measurement outcome is -1). The result is that we have transformed the n-qubit cluster state to the (n-1)-qubit cluster state, and at the same time have applied a logical Hadamard transformation, which interchanges the logical X and Z.

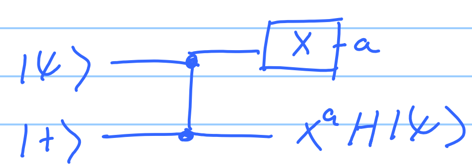
That is -- by measuring the first qubit and obtaining outcome $(-1)^a$:

$$\bar{Z}_n \longrightarrow \bar{X}_{n-1}$$

$$\bar{X}_n \longrightarrow X_2^a \bar{Z}_{n-1}$$

Here e.g. $\bar{X}_{n-1}$ means logical X on qubits 2 through n, and X_2 means X acting on qubit 2

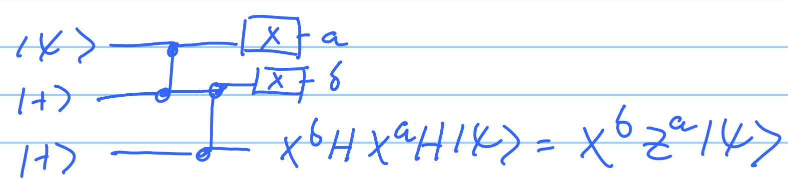
Another way to think about this is that we encode the state one step at a time as we progress along the chain from left to right. That is, we don't apply the controlled-Z gate to qubits 2 and 3 until after qubit 1 is measured. In that case, the X measurement on qubit 1 transforms a logical qubit carried by qubits 1 and 2 into a state carried by qubit 2 alone. After that we apply the controlled-Z to qubits 2 and 3 to transform to an encoded state carried by 2 and 3, etc.



$$ZI \equiv IX \longrightarrow X$$

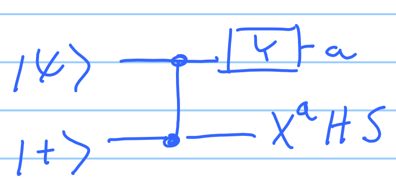
$$XZ \longrightarrow (-1)^a Z$$

If we then apply $\Lambda(Z)$ and measure qubit 2:



So ... if we measure two successive qubits, both in the X basis, we propagate the logical information two sites to the right, apart from a Pauli operator which is determined by the measurement outcomes.

We can perform other nontrivial operations on the logical qubit by choosing other measurement bases. For example, suppose that we measure Y.



$$\bar{Z} = ZI \equiv IX \longrightarrow X$$

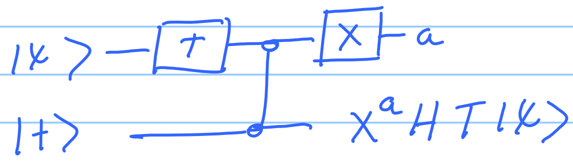
$$\bar{X} = XZ \equiv YY \longrightarrow (-1)^a Y$$

Up to a Pauli operator, we have applied HS to the input state,

where $$X \xrightarrow{S} -Y \xrightarrow{H} Y$$
$$Z \xrightarrow{S} Z \xrightarrow{H} X$$

Since S and H generate the single-qubit Clifford group, by measuring X and Y in the cluster state we can realize any Clifford transformation, up to a Pauli operator which is determined by the measurement outcomes.

For universal single-qubit computation, it would suffice to execute the T gate as well (the square root of the S gate). Consider what happens when we measure X and the input state is T |psi> rather than |psi>:



We can commute T through $\Lambda(Z)$, so this is equivalent to measuring in the rotated basis $\frac{1}{\sqrt{2}}\left(|0\rangle \pm e^{-i\pi/4}|1\rangle\right)$

However ... remember we are executing the circuit using measurements only. We are not allowed to apply Pauli operators to compensate for the Pauli operators resulting from the measurement outcomes. And if we commute a T through an X the rotation angle flips.

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \Rightarrow XTX = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & 1 \end{pmatrix} = e^{i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = e^{i\pi/4}\, T^{-1}$$

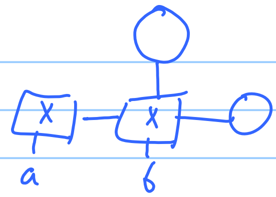
If we want to execute a circuit of H, S, and T gates, up to a known Pauli operator which is determined by the measurement outcomes, then each time we apply a T gate, we need to know whether the number of X's applied previously has been even or odd.

If it is even we measure in the basis $T|+\rangle \quad T|-\rangle$
and if it is odd we measure in basis $T^{-1}|+\rangle \quad T^{-1}|-\rangle$

In this sense, the execution of the circuit requires adaptive measurements --- each time we do a T gate, the measurement basis depends on outcomes of previous measurements.

Now we want to see how to complete our universal gate set by adding an entangling two-qubit gate, namely a CNOT gate, where we expand the cluster state to 2D. We can use two ideas already discussed: (1) Z measurements eliminate qubits from the cluster state, so by doing such measurements we can "carve out" a circuit that can be realized as a planar graph. (2) A pair of X measurements on neighboring sites just propagates a qubit forward through the graph, up to a Pauli operator determined by measurement outcomes. So a string of X measurements acts like a wire that carries a qubit. This means it suffices to understand how the entangling gate works for a three-qubit cluster state with two encoded qubits. And in fact all we have to do is measure X for one of the qubits to realize the gate.

It is a bit more convenient to consider a four-qubit cluster state instead, where we measure X for two of the qubits. In fact only one of the measurements is needed for the entangling gate, the second measurement just executes an H gate on the target qubit. But we'll consider this two step procedure because this way we actually get an encoded CNOT gate, up to a Pauli operator.



Stabilizer $\begin{matrix} & Z & \\ Z & X & Z \end{matrix}$ $\quad$ $\begin{matrix} & I & \\ I & Z & X \end{matrix}$

Logical ops $\quad X_1 = \begin{matrix} & X & \\ I & Z & I \end{matrix}$ $\quad Z_1 = \begin{matrix} & Z & \\ I & I & I \end{matrix}$

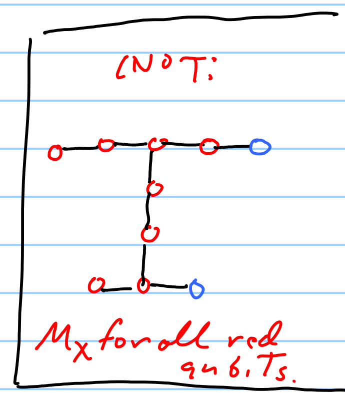
$X_2 = \begin{matrix} & I & \\ X & Z & I \end{matrix}$ $\quad Z_2 = \begin{matrix} & I & \\ Z & I & I \end{matrix}$

When we measure:

$X_1 \equiv \begin{matrix} & X & \\ I & I & X \end{matrix} \longrightarrow X X \qquad Z_1 \longrightarrow Z I$

$X_2 \equiv \begin{matrix} & I & \\ X & I & X \end{matrix} \rightarrow (-1)^a I X \qquad Z_2 \equiv \begin{matrix} & Z & \\ I & X & Z \end{matrix} \rightarrow (-1)^b Z Z$

The phases can be removed with $X_2^b Z_1^a Z_2^a$



So this is a CNOT gate, up to a two-qubit Pauli operator!

If someone is kind enough to provide us with a sufficiently cluster state, just single-qubit measurements suffice to do any quantum computation we please. The height of the cluster state we need scales with the circuit width (number of qubits), and the length scales with the circuit depth (number of time steps).

Furthermore, the cluster state does not have to be prepared all at once, it is good enough for qubits to be added to the state just before these are needed to execute gates.

The 1D cluster state as a symmetry-protected topological phase

The 1D cluster state is perhaps the simplest example of a phenomenon much studied in contemporary quantum condensed matter physics: it is a symmetry-protected topological phase (SPT phase).

First we remark that any stabilizer state or code can be interpretted as the (perhaps degenerate) ground space of a "commuting Hamiltonian". We take the Hamiltonian to be

$$H = -\sum_a \alpha_a S_a$$ where $\{S_a\}$ is the complete set of stabilizer generators and $\alpha_a > 0 \ \forall a$

We find the lowest eigenstate of H by minimizing each term separately. (If minimizing all terms simultaneously is possible we say that H is "frustration free".)This enforces S_a = 1 for each a. The ground space of H is the code. Other eigenstates of H have energy higher by at least 2 min_a (alpha_a).

For the 1D cluster state, the Hamiltonian is "geometrically local" --- this means that each term in the Hamiltonian acts on a set of qubits that are close to one another on the 1D lattice.

Let's consider an open line, where we omit the two stabilizer generators at the left and right ends of the line. Then the code space is four dimensional --- there are two encoded qubits, one localized near the left end and one near the right end. We call these "edge states" on the chain.

If the chain is n sites long, then the encoded Pauli operators acting on the left and right edges are

$$\bar{Z}_L = Z_1 \qquad \bar{X}_L = X_1 Z_2$$
$$\bar{Z}_R = Z_n \qquad \bar{X}_R = Z_{n-1} X_n$$

Here L and R stand for left and right, and e.g. Z_1 means Z acting on the first qubit in the chain.

We also note that this Hamiltonian has symmetries: there are operators which commute with the Hamiltonian, and hence map energy eigenstates (in particular ground states) to eigenstates with the same eigenvalue. There are two Pauli operators that commute with H:

$$A = X_1 X_3 X_5 \cdots$$
$$B = X_2 X_4 X_6 \cdots$$

Both commute with each $Z_{i-1} X_i Z_i$

A and B commute and both square to one. They generate a Z2 X Z2 symmetry. A acts on the odd sublattice and B acts on the even sublattice.

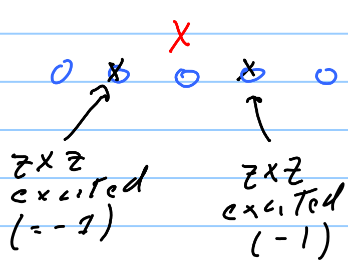
How do these symmetry operators act on the edge states? By multiplying by elements of the stabilizer, we see that acting on the code space we have

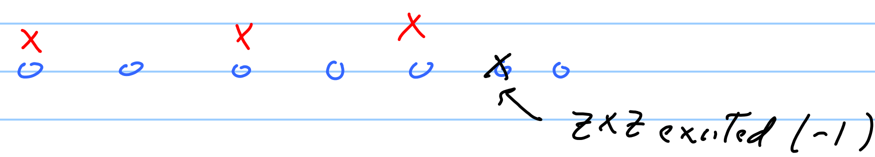
$$A = X_1 X_3 X_5 \cdots \equiv (X_1 X_3 X_5 \cdots)(Z_2 X_3 Z_4)(Z_4 X_5 Z_6) \cdots$$
$$= \begin{cases} X_1 Z_2 \cdots\cdots Z_n = X_L Z_R & \text{for } n \text{ even} \\ X_1 Z_2 \qquad Z_{n-1} X_n = X_L X_R & \text{for } n \text{ odd} \end{cases}$$

$$B = X_2 X_4 X_6 \cdots \equiv (X_2 X_4 X_6 \cdots)(Z_1 X_2 Z_3)(Z_3 X_4 Z_5) \cdots$$
$$= \begin{cases} Z_1 \cdots Z_{n-1} X_n = Z_L X_R & \text{for } n \text{ even} \\ Z_1 \cdots Z_n = Z_L Z_R & \text{for } n \text{ even} \end{cases}$$

So, acting on the ground space (but not on general states) A and B both factorize into a product of operators, one supported on the left end of the chain which acts on the left edge states, and the other supported on the right end of the chain which acts on the right edge states.

A single X acting on a site does not commute with the Hamiltonian. When it acts on an odd site (say), it creates two localized excitations ("quasiparticles") on the neighboring even sites. But a string of X's acting on successive odd sites (or successive even sites) creates an excitation only at the end of the chain.





The two symmetry operators A and B are "string" operators stretching from one end of the chain to the other. If we apply X to one site at a time, starting at the left edge and progressing toward the right edge, we view the string as the description of a process in which an excitation is created on the left, propagates across the bulk, and disappears on the right.

Note the difference between X and Z. If we apply Z to any site, an excitation is created at that site, whether of not we apply Z to other sites as well. But if we apply X's, excitations appear only at the end of the "string" of X's.

There are local operators which act on one of the two edges and preserve the ground space (the logical operators X_L and Z_L for example), but these do not respect the symmetries (they fail to commute with either A or B). For an operator to preserve the ground space and to act nontrivially on the left edge (fail to commute with X_L or Z_L), and also to have the symmetry (commute with A and B), the operator must be a nonlocal string operator, which actually acts on both edges at once.

If we consider only the action on the ground space, the symmetry operators A and B factorize into a product of two operators, each with support on the left or right edge.

$$A = A_L A_R \qquad B = B_L B_R$$

A_R and B_R act trivially on the excitation which is localized at the left edge, so it is really A_L and B_L which determine how the Z2 X Z2 symmetry acts on the left edge excitation. Now notice something interesting: While A and B commute, A_L and B_L anticommute instead --- they generate the Pauli group. Because these two operators both preserve the ground space, yet do not commute with one anothers, just the algebra of these symmetry operators is enough to inform us that the ground space must be degenerate (more than one dimensional), because acting with B_L must flip the eigenvalue of A_L:

$$A_L|\psi_0\rangle = \alpha|\psi_0\rangle \Rightarrow A_L\left(B_L|\psi_0\rangle\right) = -B_L A_L|\psi_0\rangle = -\alpha\left(B_L|\psi_0\rangle\right).$$

What is happening here? Recall what it means for a quantum system to have a symmetry group G. Each element g of G is represented by a unitary transformation U(g), and since applying first g1 and then g2 must be physically equivalent to applying the product transformation g2 g1, we must have U(g2) U(g1) = (phase) U(g2 g1).
Note that a nontrivial phase is allowed because quantum states are rays in Hilbert space. We might be able to remove the phases in the multiplication rule just by redefining the phases of { U(g) }, but if that is not possible we say the representation is projective. If fact, the Pauli matrices provide a projective representation of Z2 X Z2. What we have found is that the degenerate edge states on the left (or right) edge of the chain transform as a projective representation of the symmetry group G = Z2 X Z2 of the Hamiltonian.

We can break the degeneracy of the left edge states by adding a term to the Hamiltonian which acts on qubits 1 and 2, such as ZI (which fails to commute with A) or XZ (which fails to commute with B). Note that to lift the degeneracy it suffices to break one Z2 or the other; it is not necessary to break both. From a group theory perspective, the remaining Z2 symmetry does not suffice to enforce the degeneracy because Z2 (unlike Z2 X Z2) does not have any projective representations.

Now comes the really interesting point. Suppose we add to the Hamiltonian a small local perturbation (again a sum of geometrically local terms) which respects the Z2 X Z2 symmetry (commutes with both A and B). To be concrete, we might turn on a weak uniform "magnetic field"

$$H \longrightarrow H + H', \text{ where } \quad H' = b\sum_i X_i$$

Now the terms in the Hamiltonian are no longer mutually commuting, and diagonalizing H is not so easy. When the perturbation is weak, though, we can anticipate that

-- The low-lying states are still localized at the left and right edges.
-- The symmetry operators acting on the low-lying states still factorize into a product of operators localized at the edges.
-- The states at one edge still transform as a projective representation of Z2 X Z2.

But as we have noted, transforming as a projective representation is already enough to enforce the degeneracy of the states localized at the left (or right) edge. We conclude that the weak perturbation cannot lift the degeneracy.

This argument is not precisely correct, because when we turn on the perturbation the factorization of A and B into operators which act on just one edge is not exact. Rather A = A_L A_R, where A_L is mostly supported near the left edge, but has action on the right edge which is exponentially suppressed in n, the length of the chain. So the correct conclusion is that when we turn on the perturbation the lifting of the edge state degeneracy is exponentially small in n.

We can understand how the degeneracy is lifted by thinking about doing a perturbation expansion in powers of b (the strength of the magnetic field). Applying X to site i can create a pair of excitations at sites i-1 and i+1, or it can move an excitation from site i-1 to i+1. In a sufficiently high order in perturbation theory a process occurs in which an excitation propagates across the bulk from the left edge to the right edge, but this process is suppressed by b^{O(n)}, where n is the length of the chain. In effect, the nonlocal string order which acts on both edges arises in this order, and the quantum "tunneling" of an excitation from one edge to the other breaks the degeneracy even though the symmetry is exact. The exact degeneracy is restored in the limit of an infinitely long chain.

To justify this argument, it is important that no small energy denominators arise in the perturbation expansion --- that is, the energy cost of creating an excitation in the bulk should be a positive constant independent of n. When the pertubation is strong enough, this may no longer be true (the bulk may become "gapless") and at that stage the edge-state degeneracy can be lifted substantially.