# Ph 219b/CS 219b

**Exercises**
**Due: Wednesday 9 March 2016**

### 3.1 Positivity of quantum relative entropy

*a*) Show that $\ln x \le x-1$ for all positive real x, with equality iff $x = 1$.

*b*) The (classical) relative entropy of a probability distribution $\{p(x)\}$ relative to $\{q(x)\}$ is defined as

$$D(p \parallel q) \equiv \sum_x p(x) \left(\log p(x) - \log q(x)\right) \ . \tag{1}$$

Show that

$$D(p \parallel q) \ge 0 \ , \tag{2}$$

with equality iff the probability distributions are identical. **Hint**: Apply the inequality from ($a$) to $\ln\left(q(x)/p(x)\right)$.

*c*) The quantum relative entropy of the density operator $\boldsymbol{\rho}$ with respect to $\boldsymbol{\sigma}$ is defined as

$$D(\boldsymbol{\rho} \parallel \boldsymbol{\sigma}) = \mathrm{tr}\ \boldsymbol{\rho} \left(\log \boldsymbol{\rho} - \log \boldsymbol{\sigma}\right) \ . \tag{3}$$

Let $\{p_i\}$ denote the eigenvalues of $\boldsymbol{\rho}$ and $\{q_a\}$ denote the eigenvalues of $\boldsymbol{\sigma}$. Show that

$$D(\boldsymbol{\rho} \parallel \boldsymbol{\sigma}) = \sum_i p_i \left(\log p_i - \sum_a D_{ia} \log q_a\right) \ , \tag{4}$$

where $D_{ia}$ is a doubly stochastic matrix. Express $D_{ia}$ in terms of the eigenstates of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$. (A matrix is doubly stochastic if its entries are nonnegative real numbers, where each row and each column sums to one.)

*d*) Show that if $D_{ia}$ is doubly stochastic, then (for each $i$)

$$\log\left(\sum_a D_{ia} q_a\right) \ge \sum_a D_{ia} \log q_a \ , \tag{5}$$

with equality only if $D_{ia} = 1$ for some $a$.

*e*) Show that

$$D(\boldsymbol{\rho} \parallel \boldsymbol{\sigma}) \geq H(p \parallel r) \ , \tag{6}$$

where $r_i = \sum_a D_{ia} q_a$.

*f*) Show that $D(\boldsymbol{\rho} \parallel \boldsymbol{\sigma}) \geq 0$, with equality iff $\boldsymbol{\rho} = \boldsymbol{\sigma}$.

### 3.2 Properties of Von Neumann entropy

*a*) Use nonnegativity of quantum relative entropy to prove the *subadditivity* of Von Neumann entropy

$$H(\boldsymbol{\rho}_{AB}) \leq H(\boldsymbol{\rho}_A) + H(\boldsymbol{\rho}_B), \tag{7}$$

with equality iff $\boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$. **Hint**: Consider the relative entropy of $\boldsymbol{\rho}_{AB}$ and $\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$.

*b*) Use subadditivity to prove the concavity of the Von Neumann entropy:

$$H(\sum_x p_x \boldsymbol{\rho}_x) \geq \sum_x p_x H(\boldsymbol{\rho}_x) \ . \tag{8}$$

**Hint**: Consider

$$\boldsymbol{\rho}_{AB} = \sum_x p_x \left(\boldsymbol{\rho}_x\right)_A \otimes \left(|x\rangle\langle x|\right)_B \ , \tag{9}$$

where the states $\{|x\rangle_B\}$ are mutually orthogonal.

*c*) Use the condition

$$H(\boldsymbol{\rho}_{AB}) = H(\boldsymbol{\rho}_A) + H(\boldsymbol{\rho}_B) \quad \text{iff} \quad \boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B \tag{10}$$

to show that, if all $p_x$'s are nonzero,

$$H\left(\sum_x p_x \boldsymbol{\rho}_x\right) = \sum_x p_x H(\boldsymbol{\rho}_x) \tag{11}$$

iff all the $\boldsymbol{\rho}_x$'s are identical.

*d*) Use subadditivity to prove the triangle inequality:

$$H(\boldsymbol{\rho}_{AB}) \geq |H(\boldsymbol{\rho}_A) - H(\boldsymbol{\rho}_B)| \ . \tag{12}$$

**Hint**: Construct a "purification" of $\boldsymbol{\rho}_{AB}$ — introduce a third system $C$ and consider $|\Phi\rangle_{ABC}$ such that

$$\mathrm{tr}_C \left(|\Phi\rangle\langle\Phi|\right) = \boldsymbol{\rho}_{AB} \ ; \tag{13}$$

then use the subadditivity relations $H(\boldsymbol{\rho}_{BC}) \leq H(\boldsymbol{\rho}_B) + H(\boldsymbol{\rho}_C)$ and $H(\boldsymbol{\rho}_{AC}) \leq H(\boldsymbol{\rho}_A) + H(\boldsymbol{\rho}_C)$.

### 3.3 Separability and majorization

The hallmark of entanglement is that in an entangled state the whole is less random than its parts. But in a separable state the correlations are essentially classical and so are expected to adhere to the classical principle that the parts are less disordered than the whole. The objective of this problem is to make this expectation precise by showing that if the bipartite (mixed) state $\boldsymbol{\rho}_{AB}$ is separable, then

$$\lambda(\boldsymbol{\rho}_{AB}) \prec \lambda(\boldsymbol{\rho}_A) \ , \quad \lambda(\boldsymbol{\rho}_{AB}) \prec \lambda(\boldsymbol{\rho}_B) \ . \tag{14}$$

Here $\lambda(\boldsymbol{\rho})$ denotes the vector of eigenvalues of $\boldsymbol{\rho}$, and $\prec$ denotes majorization.

A separable state can be realized as an ensemble of pure product states, so that if $\boldsymbol{\rho}_{AB}$ is separable, it may be expressed as

$$\boldsymbol{\rho}_{AB} = \sum_a p_a \ |\psi_a\rangle\langle\psi_a| \otimes |\varphi_a\rangle\langle\varphi_a| \ . \tag{15}$$

We can also diagonalize $\boldsymbol{\rho}_{AB}$, expressing it as

$$\boldsymbol{\rho}_{AB} = \sum_j r_j |e_j\rangle\langle e_j| \ , \tag{16}$$

where $\{|e_j\rangle\}$ denotes an orthonormal basis for $AB$; then by the HJW theorem, there is a unitary matrix $V$ such that

$$\sqrt{r_j}|e_j\rangle = \sum_a V_{ja}\sqrt{p_a}|\psi_a\rangle \otimes |\varphi_a\rangle \ . \tag{17}$$

Also note that $\boldsymbol{\rho}_A$ can be diagonalized, so that

$$\boldsymbol{\rho}_A = \sum_a p_a|\psi_a\rangle\langle\psi_a| = \sum_\mu s_\mu |f_\mu\rangle\langle f_\mu| \ ; \tag{18}$$

here $\{|f_\mu\rangle\}$ denotes an orthonormal basis for $A$, and by the HJW theorem, there is a unitary matrix $U$ such that

$$\sqrt{p_a}|\psi_a\rangle = \sum_\mu U_{a\mu}\sqrt{s_\mu}|f_\mu\rangle \ . \tag{19}$$

Now show that there is a doubly stochastic matrix $D$ such that

$$r_j = \sum_\mu D_{j\mu} s_\mu \ . \tag{20}$$

That is, you must check that the entries of $D_{j\mu}$ are real and nonnegative, and that $\sum_j D_{j\mu} = 1 = \sum_\mu D_{j\mu}$. Thus we conclude that $\lambda(\boldsymbol{\rho}_{AB}) \prec \lambda(\boldsymbol{\rho}_A)$. Just by interchanging $A$ and $B$, the same argument also shows that $\lambda(\boldsymbol{\rho}_{AB}) \prec \lambda(\boldsymbol{\rho}_B)$.

**Remark**: Note that it follows from the Schur concavity of Shannon entropy that, if $\boldsymbol{\rho}_{AB}$ is separable, then the von Neumann entropy has the properties $H(AB) \geq H(A)$ and $H(AB) \geq H(B)$. Thus, for separable states, conditional entropy is nonnegative: $H(A|B) = H(AB) - H(B) \geq 0$ and $H(B|A) = H(AB) - H(A) \geq 0$. In contrast, if the state of $AB$ is an entangled pure state, then $H(AB) = 0$ and $H(B|A) = H(A|B) < 0$.

### 3.4 Entanglement of typical bipartite pure states

Suppose that a pure state is chosen at random on the bipartite system $AB$, where $d_A/d_B \ll 1$. Then with high probability the density operator on $A$ will be very nearly maximally mixed. The purpose of this problem is to derive this property.

To begin with, we will calculate the value of $\langle \mathrm{tr}\ \boldsymbol{\rho}_A^2 \rangle$, where $\langle\cdot\rangle$ denotes the average over all pure states $\{|\varphi\rangle\}$ of $AB$, and $\boldsymbol{\rho}_A = \mathrm{tr}_B\,(|\varphi\rangle\langle\varphi|)$.

*a*) It is convenient to evaluate $\mathrm{tr}\ \boldsymbol{\rho}_A^2$ using a trick. Imagine introducing a copy $A'B'$ of the system $AB$. Show that

$$\mathrm{tr}_A\ \boldsymbol{\rho}_A^2 = \mathrm{tr}_{ABA'B'}\left[(S_{AA'} \otimes I_{BB'})\,(|\varphi\rangle\langle\varphi|)_{AB} \otimes |\varphi\rangle\langle\varphi|_{A'B'})\right]\ , \tag{21}$$

where $S_{AA'}$ denotes the swap operator

$$S_{AA'} : |\varphi\rangle_A \otimes |\psi\rangle_{A'} \mapsto |\psi\rangle_A \otimes |\varphi\rangle_{A'}\ . \tag{22}$$

*b*) We wish to average the expression found in (*a*) over all pure states $|\varphi\rangle$. Rather than go into the details of how such an average is defined, I will simply assert that

$$\langle |\varphi\rangle\langle\varphi|_A \otimes |\varphi\rangle\langle\varphi|_{A'}\rangle = C\ \Pi_{AA'}\ , \tag{23}$$

where $C$ is a constant and $\Pi_{AA'}$ denotes the projector onto the subspace of $AA'$ that is *symmetric* under interchange of $A$ and $A'$. Eq. (23) can be proved using invariance properties of the average and some group representation theory, but I hope you will regard it as obvious. The state being averaged is symmetric, and the average should not distinguish any symmetric state from any other symmetric state. Express the constant $C$ in terms of the dimension $d \equiv d_A = d_{A'}$.

$c$) Use the property $\Pi_{AA'} = \frac{1}{2}\left(I_{AA'} + S_{AA'}\right)$ to evaluate the expression found in ($a$). Show that

$$\left\langle \mathrm{tr}\ \boldsymbol{\rho}_A^2 \right\rangle = \frac{d_A + d_B}{d_A d_B + 1}\ . \tag{24}$$

$d$) Now estimate the average $L^2$ distance of $\boldsymbol{\rho}_A$ from the maximally mixed density operator $\frac{1}{d_A} I_A$, where $\| \ M \ \|_2 = \sqrt{\mathrm{tr} M^\dagger M}$; show that

$$\left\langle \| \ \boldsymbol{\rho}_A - \frac{1}{d_A} I_A \ \|_2 \right\rangle \le \frac{1}{\sqrt{d_B}}\ . \tag{25}$$

**Hints**: First estimate $\left\langle \|\boldsymbol{\rho}_A - \frac{1}{d_A} I_A\|_2^2 \right\rangle$ using eq. (24) and the obvious property $\langle \boldsymbol{\rho}_A \rangle = \frac{1}{d_A} I_A$. Then show that for any nonnegative function $f$, it follows from the Cauchy-Schwarz inequality that $\left\langle \sqrt{f} \right\rangle \le \sqrt{\langle f \rangle}$, and use this property to estimate $\left\langle \| \ \boldsymbol{\rho}_A - \frac{1}{d_A} I_A \ \|_2 \right\rangle$.

$e$) Finally, estimate the average $L^1$ distance of $\boldsymbol{\rho}_A$ from the maximally mixed density operator, where $\| \ M \ \|_1 = \mathrm{tr}\ \sqrt{M^\dagger M}$. Use the Cauchy-Schwarz inequality to show that $\| \ M \ \|_1 \le \sqrt{d} \ \| \ M \ \|_2$, if $M$ is a $d \times d$ matrix, and that therefore

$$\left\langle \| \ \boldsymbol{\rho}_A - \frac{1}{d_A} I_A \ \|_1 \right\rangle \le \sqrt{\frac{d_A}{d_B}}\ . \tag{26}$$

It follows from ($d$) that the average entanglement entropy of $A$ and $B$ is close to maximal for $d_A/d_B \ll 1$: $\langle H(A) \rangle \ge \log_2 d_A - d_A/2d_B \ln 2$, though you are not asked to prove this bound. Thus, if for example $A$ is 50 qubits and $B$ is 100 qubits, the typical entropy deviates from maximal by only about $2^{-50} \approx 10^{-15}$.