Ph 219c/CS 219c

Exercises
Due: Thursday 9 March 2017

3.1 A cleaning lemma for CSS codes

In class we proved the *cleaning lemma* for stabilizer codes, which says the following: For an $[[n, k]]$ stabilizer code, let M denote a subset of the n qubits in the code block, and let M^c denote the complementary set of qubits. If P is one of the code's logical Pauli operators, we say that P can be *cleaned* on M if there is a logically equivalent Pauli operator $P' = PS$ (where S is an element of the code stabilizer) such that P' acts nontrivially only on M^c:

$$P' = I_M \otimes Q_{M^c}. \tag{1}$$

We say that P can be *supported* on M if it can be cleaned on M^c. Let $g(M)$ denote the number of independent logical Pauli operators that can be supported on M and let $g(M^c)$ denote the number of independent Pauli operators that can be supported on M^c. Then the cleaning lemma asserts that

$$g(M) + g(M^c) = 2k. \tag{2}$$

In particular, therefore, if no logical operator can be supported on M, then the complete k-qubit logical Pauli group can be supported on its complement.

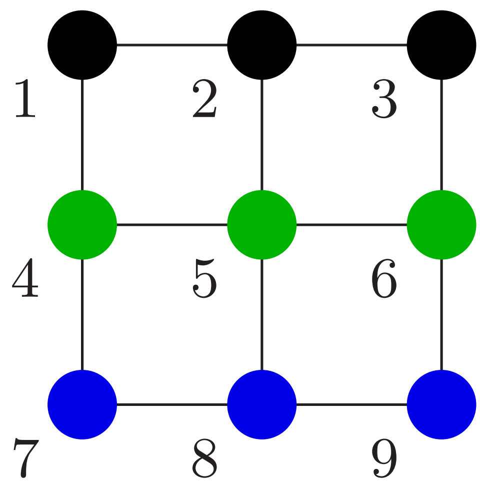
Now consider the case of an $[[n, k]]$ CSS stabilizer code, where all generators of the code stabilizer can be chosen to be either of the X type (a tensor product of X's and I's) or the Z type (a tensor product of Z's and I's); furthermore, the generators of the logical Pauli group can also be chosen to be either X type or Z type. Let $g^X(M)$ denote the number of independent X-type logical Pauli operators supported on M, and let $g^Z(M^c)$ denote the number of independent Z-type logical Pauli operators supported on M^c. Show that

$$g^X(M) + g^Z(M^c) = k. \tag{3}$$

It follows that if no X-type logical Pauli operators can be supported on M, then all Z-type logical operators can be supported on its complement.

3.2 Fault-tolerant error correction via gauge qubit measurement

Shor's [[9,1,3]] quantum code has a nice interpretation that can be appreciated by laying out the nine qubits on a 3×3 grid:



The encoded Pauli operator $\bar{X}$ can be chosen to be the tensor product of X's acting on all the qubits in a row, e.g., $\bar{X} = X_{\text{row}-1} = X_1X_2X_3$ and the encoded Pauli operator $\bar{Z}$ can be chosen to be the tensor product of Z's acting on all the qubits in column, e.g., $\bar{Z} = Z_{\text{col}-1} = Z_1Z_4Z_7$. Furthermore, the tensor product of X's on all the qubits in two rows commutes with $\bar{Z}$ and the tensor product of Z's on all the qubits in two columns commutes with $\bar{X}$. Hence we may take the stabilizer generators of the code to be

$$\begin{array}{cc} X_{\text{row}-1}X_{\text{row}-2}\ , & X_{\text{row}-2}X_{\text{row}-3}\ , \\ Z_{\text{col}-1}Z_{\text{col}-2}\ , & Z_{\text{col}-2}Z_{\text{col}-3}\ , \end{array}$$

which are mutually commuting. Note that the encoded $\bar{X}$ may be taken to be any of $X_{\text{row}-1}$, $X_{\text{row}-2}$, or $X_{\text{row}-3}$, as these differ by multiplication by an element of the stabilizer, and similarly, the encoded $\bar{Z}$ may be taken to be any of $Z_{\text{col}-1}$, $Z_{\text{col}-2}$, or $Z_{\text{col}-3}$.

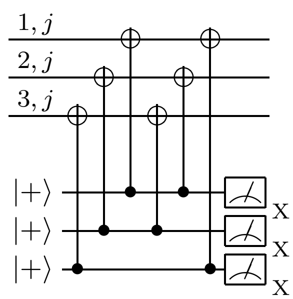
There is a (nonabelian) algebra of Pauli operators that commute with the encoded operations and with the stabilizer. This algebra includes products of neighboring X's in the same column (e.g. X_1X_4), products of neighboring Z's in the same row (e.g., Z_1Z_2), and all products of such operators. The Pauli operators in this algebra are harmless errors that preserve the stabilizer and have no effect on the encoded qubit. We will refer to the operators that commute with $\bar{X}$ and $\bar{Z}$, but that are not themselves elements of the stabilizer, as "gauge-qubit operators" (the terminology comes from an analogy with e.g. electrodynamics, where a "gauge transformation" has no effect on physical observables). A basis for the gauge-qubit operators is provided by, for example, $\{X_1X_4, Z_1Z_3, X_2X_5, Z_2Z_3, X_4X_7, Z_7Z_9, X_5X_8, Z_8Z_9\}$.

From the perspective of fault tolerance, a particularly nice feature is that (even though the gauge qubit operators are not mutually commuting) the values of the outcomes of gauge qubit measurements can be used to infer the values of the eigenvalues of the stabilizer generators.

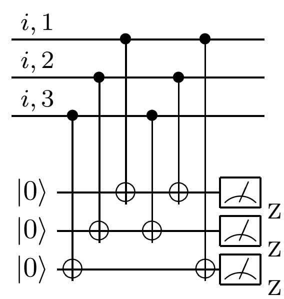
a) If gauge qubits can be measured without faults, explain how such measurements can be used to determine the error syndrome, and how the syndrome determines the appropriate recovery operation.

A procedure for measuring the weight-two gauge qubit operators in (a) column j and (b) row i is shown here:

(a)

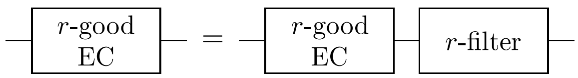


(b)

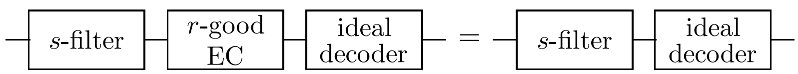


E.g., for $j = 1$, the procedure (a) measures X_1X_4, X_4X_7, and X_1X_7, and for $i = 1$, the procedure (b) measures Z_1Z_2, Z_2Z_3, and Z_1Z_3. Though X_1X_7 is not independent of the other two observables in the first column, this third redundant measurement is needed to ensure fault tolerance. Similarly, in (b) the third measurement in each row ensures fault tolerance.

Recall that, for a code that corrects t errors, a fault-tolerant error-correction circuit should have these two properties:



$(r \leq t)$



$(r + s \leq t)$

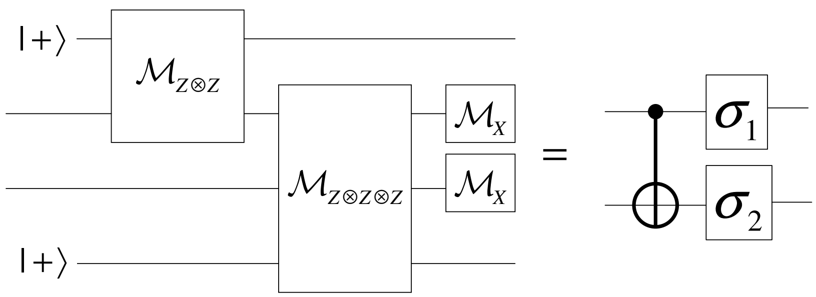
Here we use "r-good" to indicate an error correction with at most r faults, and the "s-filter" is the orthogonal projection onto the space spanned by all states that can be obtained by acting on a codeword with a Pauli operator of weight no larger than s. For the [[9,1,3]] code we have $t = 1$.

b) Suppose that the third redundant measurement in each row and column were omitted. How could a single faulty gate cause an encoded error? Show that the error correction procedure is fault tolerant (in the sense that the two properties are satisfied) when the redundant measurements are included.

c) How many locations (qubit preparations, qubit measurements, and CNOT gates) are contained in one complete cycle of fault-tolerant syndrome measurement?

3.3 A CNOT gadget constructed from measurements

Verify the following circuit identity:



Here $\mathcal{M}_\sigma$ represents measurement of the Pauli operator σ, $|+\rangle$ is the eigenstate of X with eigenvalue $+1$, and σ_1, σ_2 on the right-hand side of the equation are single-qubit Pauli operators that depend on the outcomes of the four measurements in the circuit on the left-hand side. What are σ_1 and σ_2? **Hint**: Check that Pauli operators propagate through the circuit as they do through a CNOT gate:

$$\text{CNOT}: \quad XI \to XX \ , \quad IX \to IX \ , \quad ZI \to ZI \ , \quad IZ \to ZZ \ ,$$

(where the first qubit is the control qubit and the second qubit is the target qubit of the CNOT) except for minus signs that depend on the measurement outcomes, and note that the minus signs can be removed by choosing σ_1 and σ_2 appropriately.

Though it is a bit more complicated than the measurement-based CNOT gadget constructed in class, this gadget has an advantageous feature: the Pauli operators that are measured nondestructively are Z-type operators. In some experimental settings these are easier to measure than operators that are X-type or of mixed type.