Ph219C/CS219C

Exercises
Due: Thursday 16 May 2024

3.1 Noisy superdense coding and teleportation.

a) By converting the entanglement achieved by the mother protocol into classical communication, prove the noisy superdense coding resource inequality:

$$Noisy\ SD: \quad \langle \phi_{ABE} \rangle + H(A)[q \to q] \geq I(A;B)[c \to c]. \quad (1)$$

Verify that this matches the standard noiseless superdense coding resource inequality when ϕ is a maximally entangled state of AB.

b) By converting the entanglement achieved by the mother protocol into quantum communication, prove the noisy teleportation resource inequality:

$$Noisy\ TP: \quad \langle \phi_{ABE} \rangle + I(A;B)[c \to c] \geq I_c(A\rangle B)[q \to q]. \quad (2)$$

Verify that this matches the standard noiseless teleportation resource inequality when ϕ is a maximally entangled state of AB.

3.2 Degradability of amplitude damping and erasure

The qubit amplitude damping channel $\mathcal{N}_{\text{a.d.}}^{A\to B}(p)$ discussed in §3.4.3 has the dilation $\boldsymbol{U}^{A\to BE}$ such that

$$\begin{aligned} \boldsymbol{U}: &|0\rangle_A \mapsto |0\rangle_B \otimes |0\rangle_E, \\ &|1\rangle_A \mapsto \sqrt{1-p}\ |1\rangle_B \otimes |0\rangle_E + \sqrt{p}\ |0\rangle_B \otimes |1\rangle_E; \end{aligned}$$

a qubit in its "ground state" $|0\rangle_A$ is unaffected by the channel, while a qubit in the "excited state" $|1\rangle_A$ decays to the ground state with probability p, and the decay process excites the environment. Note that $\boldsymbol{U}$ is invariant under interchange of systems B and E accompanied by transformation $p \leftrightarrow (1-p)$. Thus the channel complementary to $\mathcal{N}_{\text{a.d.}}^{A\to B}(p)$ is $\mathcal{N}_{\text{a.d.}}^{A\to E}(1-p)$.

a) Show that $\mathcal{N}_{\text{a.d.}}^{A\to B}(p)$ is degradable for $p \leq 1/2$. Therefore, the quantum capacity of the amplitude damping channel is its optimized one-shot coherent information. **Hint**: It suffices to show that

$$\mathcal{N}_{\text{a.d.}}^{A\to E}(1-p) = \mathcal{N}_{\text{a.d.}}^{B\to E}(q) \circ \mathcal{N}_{\text{a.d.}}^{A\to B}(p), \tag{3}$$

where $0 \leq q \leq 1$.

The *erasure channel* $\mathcal{N}_{\text{erase}}^{A\to B}(p)$ has the dilation $\boldsymbol{U}^{A\to BE}$ such that

$$\boldsymbol{U} : |\psi\rangle_A \mapsto \sqrt{1-p}\; |\psi\rangle_B \otimes |e\rangle_E + \sqrt{p}\; |e\rangle_B \otimes |\psi\rangle_E; \tag{4}$$

Alice's system passes either to Bob (with probability $1-p$) or to Eve (with probability p), while the other party receives the "erasure symbol" $|e\rangle$, which is orthogonal to Alice's Hilbert space. Because $\boldsymbol{U}$ is invariant under interchange of systems B and E accompanied by transformation $p \leftrightarrow (1-p)$, the channel complementary to $\mathcal{N}_{\text{erase}}^{A\to B}(p)$ is $\mathcal{N}_{\text{erase}}^{A\to E}(1-p)$.

b) Show that $\mathcal{N}_{\text{erase}}^{A\to B}(p)$ is degradable for $p \leq 1/2$. Therefore, the quantum capacity of the erasure channel is its optimized one-shot coherent information. **Hint**: It suffices to show that

$$\mathcal{N}_{\text{erase}}^{A\to E}(1-p) = \mathcal{N}_{\text{erase}}^{B\to E}(q) \circ \mathcal{N}_{\text{erase}}^{A\to B}(p), \tag{5}$$

where $0 \leq q \leq 1$.

c) Show that for $p \leq 1/2$ the quantum capacity of the erasure channel is

$$Q(\mathcal{N}_{\text{erase}}^{A\to B}(p)) = (1-2p)\log_2 d, \tag{6}$$

where A is d-dimensional, and that the capacity vanishes for $1/2 \leq p \leq 1$.

3.3 Approximate cloning and the depolarizing channel

Consider a qubit channel $\mathcal{N}^{A\to B}$ with the isometric dilation $U^{A\to BEF}$ which acts on an orthonormal basis as

$$|0\rangle_A \mapsto |\phi_0\rangle_{BEF} = \cos\theta|00\rangle_{BE}|0\rangle_F + \sin\theta|\psi^+\rangle_{BE}|1\rangle_F, \tag{7}$$

$$|1\rangle_A \mapsto |\phi_1\rangle_{BEF} = \cos\theta|11\rangle_{BE}|1\rangle_F + \sin\theta|\psi^+\rangle_{BE}|0\rangle_F, \tag{8}$$

where

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \tag{9}$$

and $0 \le \theta \le \pi/2$. Here we have split the channel's environment into two parts labeled E and F, and we have constructed the isometry to be symmetric under the interchange of B and E. Hence the channel $\mathcal{N}^{A\to B}$ obtained by tracing out EF is identical to the channel $\mathcal{N}^{A\to E}$ obtained by tracing out BF. Evidently $\mathcal{N}^{A\to B}$ is antidegradable, because we can obtain the $\mathcal{N}$ from its complementary channel $\mathcal{N}_c^{A\to EF}$ simply by tracing out F. It follows that, as for any antidegradable channel, $\mathcal{N}^{A\to B}$ has zero capacity. Putting it more prosaically, if Alice sends a quantum state to Bob via many uses of the isometry $U^{A\to BEF}$, and Bob is able to decode the state with high fidelity, then Eve receives the same output as Bob and therefore she can decode the state as well. Thus the no-cloning theorem ensures that high-fidelity decoding of the output is not possible.

a) The channel $\mathcal{N}^{A\to B}$ is actually a Pauli channel, which can be expressed as

$$\mathcal{N}(\rho) = f(\theta)\rho + g(\theta)Z\rho Z + h(\theta)\frac{I}{2}. \tag{10}$$

Find the functions f, g, and h.

b) We may choose a value $\theta = \theta_0$ such that $g(\theta_0) = 0$, in which case $\mathcal{N}$ becomes a depolarizing channel with error probability p:

$$\mathcal{N}(\rho) = \left(1 - \frac{4p}{3}\right)\rho + \left(\frac{4p}{3}\right)\frac{I}{2}. \tag{11}$$

Find this value θ_0 and the corresponding value of p. Our no-cloning argument shows that the depolarizing channel with this error probability p has zero capacity.

c) Tracing out F from the isometry $U^{A\to BEF}$, we obtain a channel $N^{A\to BE}$, which may be regarded as an approximate cloning machine. If Alice sends a pure state $|\psi\rangle$ through the channel, the marginal density operators ρ_B and ρ_E received by Bob and Eve are identical, each approximating the input state with fidelity

$$F = \langle\psi|\rho_B|\psi\rangle = \langle\psi|\rho_E|\psi\rangle. \tag{12}$$

For the value $\theta = \theta_0$ found in (*b*), compute the fidelity achieved by this approximate cloner.

d) For $\theta \neq \theta_0$, we have $g(\theta) \neq 0$, and therefore the fidelity achieved by the approximate cloner depends on the pure state input $|\psi\rangle$. In that case we may consider the worst-case fidelity $F_{\min}(\theta)$, the lowest value of F achieved for any pure state input to $\mathcal{N}^{A\to BE}$. Find the value $\theta = \theta_1$ such that this worst-case fidelity is as large as possible and compute $F_{\min}(\theta_1)$. (Be sure to recall that $g(\theta)$ changes sign at $\theta = \theta_0$.)

3.4 Proof of the decoupling inequality

In this problem we complete the derivation of the decoupling inequality sketched in §10.9.1. Equation numbers of the form (10.xxx) refer to Chapter 10 of the lecture notes.

a) Verify eq.(10.336).

To derive the expression for $\mathbb{E}_{\boldsymbol{U}}\left[\boldsymbol{M}_{AA'}(\boldsymbol{U})\right]$ in eq.(10.340), we first note that the invariance property eq.(10.325) implies that $\mathbb{E}_{\boldsymbol{U}}\left[\boldsymbol{M}_{AA'}(\boldsymbol{U})\right]$ commutes with $\boldsymbol{V}\otimes\boldsymbol{V}$ for any unitary $\boldsymbol{V}$. Therefore, by Schur's lemma, $\mathbb{E}_{\boldsymbol{U}}\left[\boldsymbol{M}_{AA'}(\boldsymbol{U})\right]$ is a weighted sum of projections onto irreducible representations of the unitary group. The tensor product of two fundamental representations of $\boldsymbol{U}(d)$ contains two irreducible representations — the symmetric and antisymmetric tensor representations. Therefore we may write

$$\mathbb{E}_{\boldsymbol{U}}\left[\boldsymbol{M}_{AA'}(\boldsymbol{U})\right] = c_{\text{sym}}\, \boldsymbol{\Pi}^{(\text{sym})}_{AA'} + c_{\text{anti}}\, \boldsymbol{\Pi}^{(\text{anti})}_{AA'}; \tag{13}$$

here $\boldsymbol{\Pi}^{(\text{sym})}_{AA'}$ is the orthogonal projector onto the subspace of AA' symmetric under the interchange of A and A', $\boldsymbol{\Pi}^{(\text{anti})}_{AA'}$ is the projector onto the antisymmetric subspace, and c_{sym}, c_{anti} are suitable constants. Note that

$$\begin{aligned}\boldsymbol{\Pi}^{(\text{sym})}_{AA'} &= \frac{1}{2}\left(\boldsymbol{I}_{AA'} + \boldsymbol{S}_{AA'}\right),\\ \boldsymbol{\Pi}^{(\text{anti})}_{AA'} &= \frac{1}{2}\left(\boldsymbol{I}_{AA'} - \boldsymbol{S}_{AA'}\right),\end{aligned} \tag{14}$$

where $\boldsymbol{S}_{AA'}$ is the swap operator, and that the symmetric and antisymmetric subspaces have dimension $\frac{1}{2}|A|\left(|A|+1\right)$ and dimension $\frac{1}{2}|A|\left(|A|-1\right)$ respectively.

Even if you are not familiar with group representation theory, you might regard eq.(13) as obvious. We may write $\boldsymbol{M}_{AA'}(\boldsymbol{U})$ as a sum of two terms, one symmetric and the other antisymmetric under the interchange of A and A'. The expectation of the symmetric part must be symmetric, and the expectation value of the antisymmetric part must be antisymmetric. Furthermore, averaging over the unitary group ensures that no symmetric state is preferred over any other.

b) To evaluate the constant c_{sym}, multiply both sides of eq.(13) by $\mathbf{\Pi}_{AA'}^{(\text{sym})}$ and take the trace of both sides, thus finding

$$c_{\text{sym}} = \frac{|A_1| + |A_2|}{|A| + 1}. \tag{15}$$

c) To evaluate the constant c_{anti}, multiply both sides of eq.(13)) by $\mathbf{\Pi}_{AA'}^{(\text{anti})}$ and take the trace of both sides, thus finding

$$c_{\text{anti}} = \frac{|A_1| - |A_2|}{|A| - 1}. \tag{16}$$

d) Using

$$c_{\boldsymbol{I}} = \frac{1}{2}\left(c_{\text{sym}} + c_{\text{anti}}\right), \quad c_{\boldsymbol{S}} = \frac{1}{2}\left(c_{\text{sym}} - c_{\text{anti}}\right) \tag{17}$$

prove eq.(10.341).