4.1 A cleaning lemma for CSS codes

In class (and in the notes) we proved the cleaning lemma for stabilizer codes, which says the following: For an $[[n, k]]$ stabilizer code, let $M$ denote a subset of the $n$ qubits in the code block, and let $M^c$ denote the complementary set of qubits. If $x$ is one of the code’s logical Pauli operators, we say that $x$ can be cleaned on $M$ if there is a logically equivalent Pauli operator $x' = xy$ (where $y$ is an element of the code stabilizer $S$) such that $x'$ acts nontrivially only on $M^c$:

$$x' = I_M \otimes Q_{M^c}. \quad (1)$$

We say that $x$ can be supported on $M$ if it can be cleaned on $M^c$. Let $g(M)$ denote the number of independent logical Pauli operators that can be supported on $M$ and let $g(M^c)$ denote the number of independent Pauli operators that can be supported on $M^c$. Then the cleaning lemma asserts that

$$g(M) + g(M^c) = 2k. \quad (2)$$

In particular, therefore, if no logical operator can be supported on $M$, then the complete $k$-qubit logical Pauli group can be supported on its complement.

Now consider the case of an $[[n, k]]$ CSS stabilizer code, where all generators of the code stabilizer can be chosen to be either of the $X$ type (a tensor product of $X$’s and $I$’s) or the $Z$ type (a tensor product of $Z$’s and $I$’s); furthermore, the generators of the logical Pauli group can also be chosen to be either $X$ type or $Z$ type. Let $g^X(M)$ denote the number of independent $X$-type logical Pauli operators supported on $M$, and let $g^Z(M^c)$ denote the number of independent $Z$-type logical Pauli operators supported on $M^c$. Show that

$$g^X(M) + g^Z(M^c) = k. \quad (3)$$

It follows that if no $X$-type logical Pauli operators can be supported on $M$, then all $Z$-type logical operators can be supported on its complement.

4.2 Bound on $D$-dimensional codes

In class (and in the notes) we proved a bound on $[[n, k, d]]$ for local stabilizer codes in 2 dimensions: $kd^2 = O(n)$, where $k$ is the number of encoded qubits, $d$ is the code distance, and $n$ is the code length.

Using similar reasoning, derive a bound on $D$-dimensional stabilizer codes of the form $kd^\alpha = O(n)$. Express $\alpha$ in terms of $D$.

4.3 Price of a code

The distance $d$ of a stabilizer code is the size of the smallest set of qubits in the code block that supports a nontrivial logical operator. In contrast, the price $p$ of a code is defined
as the size of the smallest set of qubits that supports all of the code’s logical operators. Evidently $p \geq d$.

a) What is the price of the [[7,1,3]] quantum code?

b) Use the cleaning lemma to show that $p \leq n - d + 1$.

c) Show that $p \geq k + d - 1$. **Hint:** Use this result proved in class (and in the notes): Suppose that the code block can be divided into three parts $ABC$ such that $A$ and $B$ are both correctable. Then $k \leq |C|$, where $|C|$ is the number of qubits in $C$.

d) For an $[[n, k, d]]$ stabilizer code, prove the quantum Singleton bound: $n - k \geq 2(d - 1)$.

### 4.4 Price of a local code

Using the holographic principle and the union lemma (described in class and in the notes), derive a bound on the price $p$ of a $D$-dimensional stabilizer code of the form $pd^\beta = O(n)$. Express $\beta$ in terms of $D$. 

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