Ph/CS 219C

Exercises
Due: Thursday 30 May 2024

4.1 A cleaning lemma for CSS codes

In class (and in the notes) we proved the *cleaning lemma* for stabilizer codes, which says the following: For an $[[n,k]]$ stabilizer code, let M denote a subset of the n qubits in the code block, and let M^c denote the complementary set of qubits. If x is one of the code's logical Pauli operators, we say that x can be *cleaned* on M if there is a logically equivalent Pauli operator $x' = xy$ (where y is an element of the code stabilizer S) such that x' acts nontrivially only on M^c:

$$x' = I_M \otimes Q_{M^c}. \tag{1}$$

We say that x can be *supported* on M if it can be cleaned on M^c. Let $g(M)$ denote the number of independent logical Pauli operators that can be supported on M and let $g(M^c)$ denote the number of independent Pauli operators that can be supported on M^c. Then the cleaning lemma asserts that

$$g(M) + g(M^c) = 2k. \tag{2}$$

In particular, therefore, if no logical operator can be supported on M, then the complete k-qubit logical Pauli group can be supported on its complement.

Now consider the case of an $[[n,k]]$ CSS stabilizer code, where all generators of the code stabilizer can be chosen to be either of the X type (a tensor product of X's and I's) or the Z type (a tensor product of Z's and I's); furthermore, the generators of the logical Pauli group can also be chosen to be either X type or Z type. Let $g^X(M)$ denote the number of independent X-type logical Pauli operators supported on M, and let $g^Z(M^c)$ denote the number of independent Z-type logical Pauli operators supported on M^c. Show that

$$g^X(M) + g^Z(M^c) = k. \tag{3}$$

It follows that if no X-type logical Pauli operators can be supported on M, then all Z-type logical operators can be supported on its complement.

4.2 Bound on D-dimensional codes

In class (and in the notes) we proved a bound on $[[n,k,d]]$ for local stabilizer codes in 2 dimensions: $kd^2 = O(n)$, where k is the number of encoded qubits, d is the code distance, and n is the code length.

Using similar reasoning, derive a bound on D-dimensional stabilizer codes of the form $kd^\alpha = O(n)$. Express α in terms of D.

4.3 Price of a code

The distance d of a stabilizer code is the size of the smallest set of qubits in the code block that supports a nontrivial logical operator. In contrast, the *price* p of a code is defined as the size of the smallest set of qubits that supports *all* of the code's logical operators. Evidently $p \geq d$.

a) What is the price of the [[7,1,3]] quantum code?

b) Use the cleaning lemma to show that $p \leq n - d + 1$.

c) Show that $p \geq k + d - 1$. **Hint**: Use this result proved in class (and in the notes): Suppose that the code block can be divided into three parts ABC such that A and B are both correctable. Then $k \leq |C|$, where $|C|$ is the number of qubits in C.

d) For an $[[n, k, d]]$ stabilizer code, prove the *quantum Singleton bound*: $n - k \geq 2(d - 1)$.

4.4 Price of a local code

Using the holographic principle and the union lemma (described in class and in the notes), derive a bound on the price p of a D-dimensional stabilizer code of the form $pd^{\beta} = O(n)$. Express β in terms of D.