Ph 219b/CS 219b

Exercises
Due: Thursday 5 December 2019

4.1 Estimating the trace of a unitary matrix

Recall that using an oracle that applies the conditional unitary $\Lambda(U)$,

$$\begin{aligned}\Lambda(U): \qquad & |0\rangle \otimes |\psi\rangle \mapsto |0\rangle \otimes |\psi\rangle \ , \\ & |1\rangle \otimes |\psi\rangle \mapsto |1\rangle \otimes U|\psi\rangle \end{aligned} \tag{1}$$

(where U is a unitary transformation acting on n qubits), we can measure the eigenvalues of U. If the state $|\psi\rangle$ is the eigenstate $|\lambda\rangle$ of U with eigenvalue $\lambda = \exp(2\pi i\phi)$, then by querying the oracle k times, we can determine ϕ to accuracy $O(1/\sqrt{k})$.

But suppose that we replace the pure state $|\psi\rangle$ in eq. (1) by the maximally mixed state of n qubits, $\rho = I/2^n$.

a) Show that, with k queries, we can estimate both the real part and the imaginary part of $\mathrm{tr}\,(U)\,/2^n$, the normalized trace of U, to accuracy $O(1/\sqrt{k})$.

b) Given a polynomial-size quantum circuit, the problem of estimating to fixed accuracy the normalized trace of the unitary transformation realized by the circuit is believed to be a hard problem classically. Explain how this problem can be solved efficiently with a quantum computer.

The initial state needed for each query consists of one qubit in the pure state $|0\rangle$ and n qubits in the maximally mixed state. Surprisingly, then, the initial state of the computer that we require to run this (apparently) powerful quantum algorithm contains only a constant number of "clean" qubits, and $O(n)$ very noisy qubits.

4.2 A generalization of Simon's problem

Simon's problem is a hidden subgroup problem with $G = Z_2^n$ and $H = Z_2 = \{0, a\}$. Consider instead the case where $H = Z_2^k$, with generator set $\{a_i, i = 1, 2, 3, \ldots, k\}$. That is, suppose an oracle evaluates a function

$$f : \{0,1\}^n \to \{0,1\}^{n-k} \ , \tag{2}$$

where we are promised that f is 2^k-to-1 such that

$$f(x) = f(x \oplus a_i) \tag{3}$$

for $i = 1, 2, 3, \ldots, k$ (here $\oplus$ denotes bitwise addition modulo 2). Since the number of cosets of H in G is smaller, we can expect that the hidden subgroup is easier to find for this problem than in Simon's ($k = 1$) case.

Find an algorithm using $n - k$ quantum queries that identifies the k generators of H, and show that the success probability of the algorithm is greater than $1/4$.

4.3 Finding a collision

Suppose that a black box evaluates a function

$$f : \{0,1\}^n \to \{0,1\}^{n-1} \ . \tag{4}$$

We are promised that the function is 2-to-1, and we are to find a "collision" – values x and y such that $f(x) = f(y)$. This problem is harder than Simon's problem, because we are not promised that the function is periodic. Let $N = 2^n$.

a) Describe a randomized classical algorithm that requires SPACE = $O(\sqrt{N})$ and that succeeds in finding a collision with high probability in $O(\sqrt{N})$ queries of the black box.

b) Now suppose that only SPACE = $O(N^{1/3})$ is available. Describe a randomized classical algorithm that finds a collision with high probability in $O(N^{2/3})$ queries.

c) Show that Grover's exhaustive search algorithm can be used to find a collision in $O(\sqrt{N})$ quantum queries, using SPACE = $O(1)$.

d) Describe a quantum algorithm that uses SPACE = $O(M)$ and finds a collision in $O(M) + O(\sqrt{N/M})$ quantum queries. [**Hint**: First query the box M times to learn the value of $f(x)$ for M arguments $\{x_1, x_2, \ldots, x_M\}$, then search for y such that $f(y) = f(x_i)$ for some x_i.] Thus, if M is chosen to optimize the number of queries, the quantum algorithm uses SPACE = $O(N^{1/3})$ and $O(N^{1/3})$ quantum queries.

4.4 Quantum counting

A black box computes a function

$$f : \{0,1\}^n \to \{0,1\} \ , \tag{5}$$

which can be represented by a binary string

$$X = X_{N-1}X_{N-2}\cdots X_1 X_0 \ , \tag{6}$$

where $X_i = f(i)$ and $N = 2^n$. Our goal is to count the number r of states "marked" by the box; that is, to determine the Hamming weight $r = |X|$ of X. We can devise a quantum algorithm that counts the marked states by combining Grover's exhaustive search with the quantum Fourier transform.

a) The black box performs an $(n{+}1)$-qubit unitary transformation U_f which acts on a basis according to

$$U_f\left(|x\rangle \otimes |y\rangle\right) = |x\rangle \otimes |y \oplus f(x)\rangle. \tag{7}$$

If the last qubit is set to the state $|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$, then the box applies the unitary transformation $\tilde{U}_f$ to the first n qubits, where

$$\tilde{U}_f|x\rangle = (-1)^{f(x)}|x\rangle. \tag{8}$$

Explain how to use the box and Hadamard gates to perform $\Lambda(\tilde{U}_f)$, the unitary $\tilde{U}_f$ conditioned on the value of a control qubit.

b) Let

$$|\Psi_X\rangle = \frac{1}{\sqrt{r}} \sum_{j:X_j=1} |j\rangle \tag{9}$$

denote the uniform superposition of the marked states, and let U_{Grover} denote the "Grover iteration," which performs a rotation by the angle 2θ in the plane spanned by $|\Psi_X\rangle$ and

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N} |j\rangle \ , \tag{10}$$

where

$$\sin\theta = \langle s|\Psi_X\rangle = \sqrt{\frac{r}{N}} \ . \tag{11}$$

Consider a unitary transformation

$$V : |t\rangle \otimes |\Phi\rangle \to |t\rangle \otimes U_{\text{Grover}}^{t} |\Phi\rangle \tag{12}$$

that reads a counter register taking values $t \in \{0, 1, 2, \ldots, T-1\}$ (where $T = 2^m$), and then applies U_{Grover} t times. Explain how V can be implemented, calling the oracle $T-1$ times. [**Hint:** Use the binary expansion $t = \sum_{k=0}^{m-1} t_k 2^k$ and the conditional oracle call from (a).]

c) Suppose that $r \ll N$. Show that, by applying V, performing the quantum Fourier transform on the counter register, and then measuring the counter register, we can determine θ to accuracy $O(1/T)$, and hence we can find r with high success probability in $T = O(\sqrt{rN})$ queries. Compare to the best classical protocol.