Ph 219c/CS 219c

Exercises
Due: Thursday 25 May 2017

6.1 Positivity of quantum relative entropy

a) Show that $\ln x \leq x-1$ for all positive real x, with equality iff $x = 1$.

b) The (classical) relative entropy of a probability distribution $\{p(x)\}$ relative to $\{q(x)\}$ is defined as

$$D(p \parallel q) \equiv \sum_x p(x) \left(\log p(x) - \log q(x)\right) \ . \tag{1}$$

Show that

$$D(p \parallel q) \geq 0 \ , \tag{2}$$

with equality iff the probability distributions are identical. **Hint**: Apply the inequality from (a) to $\ln\left(q(x)/p(x)\right)$.

c) The quantum relative entropy of the density operator $\boldsymbol{\rho}$ with respect to $\boldsymbol{\sigma}$ is defined as

$$D(\boldsymbol{\rho} \parallel \boldsymbol{\sigma}) = \mathrm{tr}\ \boldsymbol{\rho} \left(\log \boldsymbol{\rho} - \log \boldsymbol{\sigma}\right) \ . \tag{3}$$

Let $\{p_i\}$ denote the eigenvalues of $\boldsymbol{\rho}$ and $\{q_a\}$ denote the eigenvalues of $\boldsymbol{\sigma}$. Show that

$$D(\boldsymbol{\rho} \parallel \boldsymbol{\sigma}) = \sum_i p_i \left(\log p_i - \sum_a D_{ia} \log q_a\right) \ , \tag{4}$$

where D_{ia} is a doubly stochastic matrix. Express D_{ia} in terms of the eigenstates of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$. (A matrix is doubly stochastic if its entries are nonnegative real numbers, where each row and each column sums to one.)

d) Show that if D_{ia} is doubly stochastic, then (for each i)

$$\log\left(\sum_a D_{ia} q_a\right) \geq \sum_a D_{ia} \log q_a \ , \tag{5}$$

with equality only if $D_{ia} = 1$ for some a.

e) Show that

$$D(\boldsymbol{\rho} \parallel \boldsymbol{\sigma}) \geq D(p \parallel r) \ , \tag{6}$$

where $r_i = \sum_a D_{ia} q_a$.

f) Show that $D(\boldsymbol{\rho} \parallel \boldsymbol{\sigma}) \geq 0$, with equality iff $\boldsymbol{\rho} = \boldsymbol{\sigma}$.

6.2 Properties of Von Neumann entropy

a) Use nonnegativity of quantum relative entropy to prove the *subadditivity* of Von Neumann entropy

$$H(\boldsymbol{\rho}_{AB}) \leq H(\boldsymbol{\rho}_A) + H(\boldsymbol{\rho}_B), \tag{7}$$

with equality iff $\boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$. **Hint**: Consider the relative entropy of $\boldsymbol{\rho}_{AB}$ and $\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$.

b) Use subadditivity to prove the concavity of the Von Neumann entropy:

$$H(\sum_x p_x \boldsymbol{\rho}_x) \geq \sum_x p_x H(\boldsymbol{\rho}_x) \ . \tag{8}$$

Hint: Consider

$$\boldsymbol{\rho}_{AB} = \sum_x p_x \, (\boldsymbol{\rho}_x)_A \otimes (|x\rangle\langle x|)_B \ \ , \tag{9}$$

where the states $\{|x\rangle_B\}$ are mutually orthogonal.

c) Use the condition

$$H(\boldsymbol{\rho}_{AB}) = H(\boldsymbol{\rho}_A) + H(\boldsymbol{\rho}_B) \quad \text{iff} \quad \boldsymbol{\rho}_{AB} = \boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B \tag{10}$$

to show that, if all p_x's are nonzero,

$$H\left(\sum_x p_x \boldsymbol{\rho}_x\right) = \sum_x p_x H(\boldsymbol{\rho}_x) \tag{11}$$

iff all the $\boldsymbol{\rho}_x$'s are identical.

6.3 Monotonicity of quantum relative entropy

Quantum relative entropy has a property called *monotonicity*:

$$D(\boldsymbol{\rho}_A \| \boldsymbol{\sigma}_A) \leq D(\boldsymbol{\rho}_{AB} \| \boldsymbol{\sigma}_{AB}); \tag{12}$$

The relative entropy of two density operators on a system AB cannot be less than the induced relative entropy on the subsystem A.

a) Use monotonicity of quantum relative entropy to prove the strong subadditivity property of Von Neumann entropy. **Hint**: On a tripartite system ABC, consider the relative entropy of $\boldsymbol{\rho}_{ABC}$ and $\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_{BC}$.

b) Use monotonicity of quantum relative entropy to show that the action of a quantum channel $\mathcal{N}$ cannot increase relative entropy:

$$D(\mathcal{N}(\boldsymbol{\rho}) \| \mathcal{N}(\boldsymbol{\sigma}) \leq D(\boldsymbol{\rho} \| \boldsymbol{\sigma}), \tag{13}$$

Hint: Recall that any quantum channel has an isometric dilation.

6.4 The first law of Von Neumann entropy

Writing the density operator in terms of its *modular Hamiltonian* $\boldsymbol{K}$ as in §10.2.6,

$$\boldsymbol{\rho} = \frac{e^{-\boldsymbol{K}}}{\mathrm{tr}\left(e^{-\boldsymbol{K}}\right)}, \tag{14}$$

consider how the entropy $S(\boldsymbol{\rho}) = -\mathrm{tr}\,(\boldsymbol{\rho} \ln \boldsymbol{\rho})$ changes when the density operator is perturbed slightly:

$$\boldsymbol{\rho} \to \boldsymbol{\rho}' = \boldsymbol{\rho} + \delta\boldsymbol{\rho}. \tag{15}$$

Since $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are both normalized density operators, we have $\mathrm{tr}\,(\delta\boldsymbol{\rho}) = 0$. Show that

$$S(\boldsymbol{\rho}') - S(\boldsymbol{\rho}) = \mathrm{tr}\left(\boldsymbol{\rho}'\boldsymbol{K}\right) - \mathrm{tr}\,(\boldsymbol{\rho}\boldsymbol{K}) + O\left((\delta\boldsymbol{\rho})^2\right); \tag{16}$$

that is,

$$\delta S = \delta\langle \boldsymbol{K} \rangle \tag{17}$$

to first order in the small change in $\boldsymbol{\rho}$. This statement generalizes the first law of thermodynamics; for the case of a thermal density operator with $\boldsymbol{K} = T^{-1}\boldsymbol{H}$ (where $\boldsymbol{H}$ is the Hamiltonian and T is the temperature), it becomes the more familiar statement

$$\delta E = \delta\langle \boldsymbol{H} \rangle = T\delta S. \tag{18}$$

6.5 Quantum Singleton bound

As noted in chapter 7, an $[[n,k,d]]$ quantum error-correcting code (k protected qudits in a block of n qudits, with code distance d) must obey the constraint

$$n - k \geq 2(d-1), \tag{19}$$

the *quantum Singleton bound*. This bound is actually a corollary of a stronger statement which you will prove in this exercise.

Suppose that in the pure state ϕ_{RA} the reference system R is maximally entangled with a code subspace of A, and that E_1 and E_2 are two disjoint correctable subsystems of system A (erasure of either E_1 or E_2 can be corrected). You are to show that

$$\log|A| - \log|R| \geq \log|E_1| + \log|E_2|. \tag{20}$$

Let E^c denote the subsystem of A complementary to E_1E_2, so that $A = E^cE_1E_2$.

a) Recalling the error correction conditions $\boldsymbol{\rho}_{RE_1} = \boldsymbol{\rho}_R \otimes \boldsymbol{\rho}_{E_1}$ and $\boldsymbol{\rho}_{RE_2} = \boldsymbol{\rho}_R \otimes \boldsymbol{\rho}_{E_2}$, show that $\phi_{RE^cE_1E_2}$ has the property

$$H(R) = H(E^c) - \frac{1}{2}I(E^c;E_1) - \frac{1}{2}I(E^c;E_2). \tag{21}$$

b) Show that eq.(21) implies eq.(20).