Ph 219c/CS 219c

Exercises
Due: Thursday 7 June 2018

1 Fibonacci anyons I

For the Fibonacci anyon model there is a trivial label (denoted 0) and a nontrivial label (denoted 1); the fusion rule is

$$1 \times 1 = 0 + 1. \tag{1}$$

As discussed in Sec. 9.16 of the lecture notes, for this model there are only two F-matrices that arise, which we will denote as

$$F_{0111} \equiv F_0 \ , \quad F_{1111} \equiv F_1 \ . \tag{2}$$

F_0 is really the 1×1 matrix

$$(F_0)_a^b = \delta_a^1 \delta_1^b \ , \tag{3}$$

while F_1 is a 2×2 matrix. The pentagon equation becomes

$$(F_c)_a^d \, (F_a)_b^c = \sum_e (F_d)_e^c \, (F_e)_b^d \, (F_b)_a^e \ . \tag{4}$$

Show that the general solution for $F \equiv F_1$ is

$$F = \begin{pmatrix} \tau & e^{i\phi}\sqrt{\tau} \\ e^{-i\phi}\sqrt{\tau} & -\tau \end{pmatrix} \ , \tag{5}$$

where $e^{i\phi}$ is an arbitrary phase (which we can set to 1 with a suitable phase convention), and $\tau = \left(\sqrt{5} - 1\right)/2 \approx .618$, which satisfies

$$\tau^2 + \tau = 1 \ . \tag{6}$$

2 Fibonacci anyons II

The 2×2 R-matrix that describes a counterclockwise exchange of two Fibonacci anyons has two eigenvalues — R^0 for the case where the total charge of the pair of anyons is trivial, and R^1 for the case where the total charge is nontrivial. The hexagon equation becomes

$$R^c\,(F)^c_a\,R^a = (F)^c_0\,(F)^0_a + (F)^c_1\,R^1\,(F)^1_a\ . \tag{7}$$

Using the expression for F found by solving the pentagon equation (with $e^{i\phi}$ set equal to 1), solve the hexagon equation for R, finding

$$R = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & e^{-3\pi i/5} \end{pmatrix}\ ,\quad F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}\ . \tag{8}$$

The only other solution for R is the complex conjugate of this one; this second solution really describes the same model, but with clockwise and counterclockwise braiding interchanged. Therefore, an anyon model with the Fibonacci fusion rule really *does* exist, and it is essentially unique.

3 Ising anyons

The fusion rule for two Ising anyons is

$$\sigma \times \sigma = 1 + \psi, \tag{9}$$

where here 1 denotes the trivial charge and ψ denotes the fermion. By solving the pentagon and hexagon equations we can find the 2×2 R and F matrices

$$R \equiv R_{\sigma\sigma} = e^{-i\theta}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\ ,\quad F \equiv F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{10}$$

where $e^{i\theta} = e^{i\pi/8}$ is the topological spin of the σ particle. (This is actually one of eight possible solutions.)

Four σ anyons can fuse with trivial total charge in two distinct ways, and therefore can encode a qubit. Suppose the anyons are lined up in order 1234, numbered from left to right; in the standard basis state $|0\rangle$, anyons 1 and 2 fuse to yield total charge 1, while in the standard basis state $|1\rangle$, anyons 1 and 2 fuse to yield total charge ψ. Acting on this standard basis, the braid group generator σ_1 (counterclockwise exchange of particles 1 and 2) is represented by R and the generator σ_2 (counterclockwise exchange of particles 2 and 3) is represented by $B = FRF$. Verify that R and B satisfy the Yang-Baxter relation.