Ph 219a/CS 219a

Exercises
Due: Wednesday 29 October 2008

1.1 Alice does Bob a favor

Alice, in Anaheim, and Bob, in Boston, share a bipartite pure state $|\Psi\rangle$, which can be expressed in the Schmidt form

$$|\Psi\rangle = \sum_i \sqrt{p_i}\, |\alpha_i\rangle \otimes |\beta_i\rangle \ , \tag{1}$$

where $\{|\alpha_i\rangle\}$ is an orthonormal basis for Alice's system A, $\{|\beta_i\rangle\}$ is an orthonormal basis for Bob's system B, and the $\{p_i\}$ are nonnegative real numbers summing to 1. Bob is supposed to perform a complete orthogonal local measurement on B, characterized by the set of projectors $\{E_a^B\}$ — if the measurement outcome is a, then Bob's measurement prepares the state

$$|\Psi\rangle \mapsto |\Psi_a\rangle = \frac{\left(I \otimes E_a^B\right)|\Psi\rangle}{\langle\Psi|\left(I \otimes E_a^B\right)|\Psi\rangle^{1/2}} \ . \tag{2}$$

$|\Psi_a\rangle$ can also be expressed in the Schmidt form if we choose appropriate orthonormal bases for A and B that depend on the measurement outcome. The new Schmidt basis elements can be written as

$$|\alpha'_{a,i}\rangle = U_a^A|\alpha_i\rangle \ , \quad |\beta'_{a,i}\rangle = U_a^B|\beta_i\rangle \ , \tag{3}$$

where U_a^A, U_a^B are unitary.

Unfortunately, Bob's measurement apparatus is broken, though he still has the ability to perform local unitary transformations on B. Show that Alice can help Bob out by performing a measurement that is "locally equivalent" to Bob's. That is, there are orthogonal projectors $\{E_a^A\}$ and unitary transformations V_a^A, V_a^B such that

$$|\Psi_a\rangle = V_a^A \otimes V_a^B \left(\frac{\left(E_a^A \otimes I\right)|\Psi\rangle}{\langle\Psi|\left(E_a^A \otimes I\right)|\Psi\rangle^{1/2}}\right) \tag{4}$$

for each a, and furthermore, both Alice's measurement and Bob's yield outcome a with the same probability. This means that instead of Bob doing the measurement, the same effect can be achieved if Alice measures instead, tells Bob the outcome, and both Alice and Bob perform the appropriate unitary transformations. Construct E_a^A (this is most conveniently done by expressing both E_a^A and E_a^B in the Schmidt bases for $|\Psi\rangle$) and express V_a^A and V_a^B in terms of U_a^A and U_a^B.

Remark: This result shows that for any protocol involving local operations and "two-way" classical communication (2-LOCC) that transforms an initial bipartite pure state to a final bipartite pure state, the same transformation can be achieved by a "one-way" (1-LOCC) protocol in which all classical communication is from Alice to Bob (the *Lo-Popescu Theorem*). In a two-way LOCC protocol, Alice and Bob take turns manipulating the state for some finite (but arbitrarily large) number of rounds. In each round, one party or the other performs a measurement on her/his local system and broadcasts the outcome to the other party. Either party might use a local "ancilla" system in performing the measurement, but we may include all ancillas used during the protocol in the bipartite pure state $|\Psi\rangle$. Though a party might discard information about the measurement outcome, or fail to broadcast the information to the other party, we are entitled to imagine that the complete information about the outcomes is known to both parties at each step (incomplete information is just equivalent to the special case in which the parties choose not to use all the information). Thus the state is pure after each step.

The solution to the exercise shows that a round of a 2-LOCC protocol in which Bob measures can be simulated by an operation performed by Alice and a local unitary applied by Bob. Thus, we can allow Alice to perform all the measurements herself. When she is through she sends all the outcomes to Bob, and he can apply the necessary product of unitary transformations to complete the protocol.

1.2 What probability distributions are consistent with a mixed state?

A density operator ρ, expressed in the orthonormal basis $\{|\alpha_i\rangle\}$ that diagonalizes it, is

$$\rho = \sum_i p_i |\alpha_i\rangle\langle\alpha_i| \ . \tag{5}$$

We would like to realize this density operator as an ensemble of pure states $\{|\varphi_\mu\rangle\}$, where $|\varphi_\mu\rangle$ is prepared with a specified probability q_μ. This preparation is possible if the $|\varphi_\mu\rangle$'s can be chosen so that

$$\rho = \sum_\mu q_\mu |\varphi_\mu\rangle\langle\varphi_\mu| \ . \tag{6}$$

We say that a probability vector q (a vector whose components are nonnegative real numbers that sum to 1) is *majorized* by a probability vector p (denoted $q \prec p$), if there exists a *doubly stochastic* matrix D such that

$$q_\mu = \sum_i D_{\mu i}\ p_i \ . \tag{7}$$

A matrix is doubly stochastic if its entries are nonnegative real numbers such that $\sum_\mu D_{\mu i} = \sum_i D_{\mu i} = 1$. That the columns sum to one assures that D maps probability vectors to probability vectors (*i.e.*, is *stochastic*). That the rows sum to one assures that D maps the uniform distribution to itself. Applied repeatedly, D takes any input distribution closer and closer to the uniform distribution (unless D is a permutation, with one nonzero entry in each row and column). Thus we can view majorization as a partial order on probability vectors such that $q \prec p$ means that q is more nearly uniform than p (or equally close to uniform, in the case where D is a permutation).

Show that normalized pure states $\{|\varphi_\mu\rangle\}$ exist such that eq. (6) is satisfied if and only if $q \prec p$, where p is the vector of eigenvalues of ρ.

Hint: Recall that, according to the *Hughston-Jozsa-Wootters Theorem*, if eq. (5) and eq. (6) are both satisfied then there is a unitary matrix $V_{\mu i}$ such that

$$\sqrt{q_\mu}\ |\varphi_\mu\rangle = \sum_i \sqrt{p_i}\ V_{\mu i}|\alpha_i\rangle \ . \tag{8}$$

You may also use (but need not prove) *Horn's Lemma*: if $q \prec p$, then there exists a unitary (in fact, orthogonal) matrix $U_{\mu i}$ such that $q = Dp$ and $D_{\mu i} = |U_{\mu i}|^2$.

1.3 What transformations are possible for bipartite pure states?

Alice and Bob share a bipartite pure state $|\Psi\rangle$. Using a 2-LOCC protocol, they wish to transform it to another bipartite pure state

$|\Phi\rangle$. Furthermore, the protocol must be *deterministic* — the state $|\Phi\rangle$ is obtained with probability one irrespective of the outcomes of the measurements that Alice and Bob perform.

Suppose that these initial and final states have Schmidt decompositions

$$|\Psi\rangle = \sum_i \sqrt{(p_\Psi)_i}\ |\alpha_i\rangle \otimes |\beta_i\rangle \ , \quad |\Phi\rangle = \sum_i \sqrt{(p_\Phi)_i}\ |\alpha_i'\rangle \otimes |\beta_i'\rangle \ . \quad (9)$$

Show that if the deterministic transformation $|\Psi\rangle \mapsto |\Phi\rangle$ is possible, then $p_\Psi \prec p_\Phi$.

Hints: Using the Lo-Popescu Theorem from Exercise 1.1, we can reduce the 2-LOCC to an equivalent 1-LOCC. That is, if the deterministic transformation is possible, then there is a generalized measurement that can be applied by Alice, and an operation depending on Alice's measurement outcome that can be applied by Bob, such that for each possible measurement outcome Alice's measurement followed by Bob's operation maps $|\Psi\rangle$ to $|\Phi\rangle$. Recall that a generalized measurement is defined by a set of operators $\{M_\mu\}$ such that $\sum_\mu M_\mu^\dagger M_\mu = I$, and that the action of the measurement on a pure state $|\psi\rangle$ if outcome μ occurs is

$$|\psi\rangle \mapsto \frac{M_\mu|\psi\rangle}{\sqrt{\langle\psi|M_\mu^\dagger M_\mu|\psi\rangle}} \ . \quad (10)$$

Think about how the 1-LOCC protocol transforms Alice's density operator. You might want to use the *polar decomposition*: a matrix A can be expressed as $\sqrt{AA^\dagger}\ U$, where U is unitary.

Remark: The converse is also true. Thus majorization provides the necessary and sufficient condition for the deterministic transformation of one bipartite pure state to another (*Nielsen's Theorem*). In this respect, majorization defines a partial order on bipartite pure states such that we may say that $|\Psi\rangle$ is no less entangled than $|\Phi\rangle$ if $p_\Psi \prec p_\Phi$.

1.4 How far apart are two quantum states?

Consider two quantum states described by density operators ρ and $\tilde{\rho}$ in an N-dimensional Hilbert space, and consider the complete orthogonal measurement $\{E_a, a = 1, 2, 3, \ldots N\}$, where the E_a's are onedimensional projectors satisfying

$$\sum_{a=1}^{N} E_a = I \ . \tag{11}$$

When the measurement is performed, outcome a occurs with probability $p_a = \mathrm{tr}\ \rho E_a$ if the state is ρ and with probability $\tilde{p}_a = \mathrm{tr}\ \tilde{\rho} E_a$ if the state is $\tilde{\rho}$.

The L^1 *distance* between the two probability distributions is defined as

$$d(p, \tilde{p}) \equiv \|p - \tilde{p}\|_1 \equiv \frac{1}{2} \sum_{a=1}^{N} |p_a - \tilde{p}_a| \ ; \tag{12}$$

this distance is zero if the two distributions are identical, and attains its maximum value one if the two distributions have support on disjoint sets.

a) Show that

$$d(p, \tilde{p}) \leq \frac{1}{2} \sum_{i=1}^{N} |\lambda_i| \tag{13}$$

where the λ_i's are the eigenvalues of the Hermitian operator $\rho - \tilde{\rho}$. **Hint**: Working in the basis in which $\rho - \tilde{\rho}$ is diagonal, find an expression for $|p_a - \tilde{p}_a|$, and then find an upper bound on $|p_a - \tilde{p}_a|$. Finally, use the completeness property eq. (11) to bound $d(p, \tilde{p})$.

b) Find a choice for the orthogonal projector $\{E_a\}$ that saturates the upper bound eq. (13).

Define a distance $d(\rho, \tilde{\rho})$ between density operators as the maximal L^1 distance between the corresponding probability distributions that can be achieved by any orthogonal measurement. From the results of (a) and (b), we have found that

$$d(\rho, \tilde{\rho}) = \frac{1}{2} \sum_{i=1}^{N} |\lambda_i| \ . \tag{14}$$

c) The L^1 *norm* $\|A\|_1$ of an operator A is defined as

$$\|A\|_1 \equiv \mathrm{tr}\left[(AA^\dagger)^{1/2}\right] \ . \tag{15}$$

How can the distance $d(\rho, \tilde{\rho})$ be expressed as the L^1 norm of an operator?

Now suppose that the states ρ and $\tilde{\rho}$ are pure states $\rho = |\psi\rangle\langle\psi|$ and $\tilde{\rho} = |\tilde{\psi}\rangle\langle\tilde{\psi}|$. If we adopt a suitable basis in the space spanned by the two vectors, and appropriate phase conventions, then these vectors can be expressed as

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} , \quad |\tilde{\psi}\rangle = \begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix} , \tag{16}$$

where $\langle\psi|\tilde{\psi}\rangle = \sin\theta$.

d) Express the distance $d(\rho, \tilde{\rho})$ in terms of the angle θ.

e) Express $\| \, |\psi\rangle - |\tilde{\psi}\rangle\|^2$ (where $\| \cdot \|$ denotes the Hilbert space norm) in terms of θ, and by comparing with the result of (d), derive the bound

$$d(|\psi\rangle\langle\psi|, |\tilde{\psi}\rangle\langle\tilde{\psi}|) \leq \| \, |\psi\rangle - |\tilde{\psi}\rangle\| \ . \tag{17}$$

f) Bob thinks that the norm $\| \, |\psi\rangle - |\tilde{\psi}\rangle\|$ should be a good measure of the distinguishability of the pure quantum states ρ and $\tilde{\rho}$. Explain why Bob is wrong. **Hint**: Remember that quantum states are *rays*.