Ph 219/CS 219

Exercises
Due: Friday 3 November 2006

2.1 Fidelity

We saw in Exercise 1.1 that the trace norm $\|\rho-\tilde{\rho}\|_{\rm tr}$ provides a useful measure of the distinguishability of the states ρ and $\tilde{\rho}$. Another useful measure of distinguishability is the *fidelity* $F(\rho,\tilde{\rho})$, which is defined as

$$F(\rho,\tilde{\rho}) \equiv \| \tilde{\rho}^{\frac{1}{2}}\rho^{\frac{1}{2}} \|_{\rm tr}^2 = \left({\rm tr}\ \sqrt{\rho^{\frac{1}{2}}\tilde{\rho}\rho^{\frac{1}{2}}}\ \right)^2 . \qquad (1)$$

(Some authors use the name "fidelity" for the square root of this quantity.) The fidelity is nonnegative, vanishes if ρ and $\tilde{\rho}$ have support on mutually orthogonal subspaces, and attains its maximum value 1 if and only if the two states are identical.

a) The fidelity $F(\rho,\tilde{\rho})$ is actually symmetric in its two arguments, although the symmetry is not manifest in eq. (1). To demonstrate the symmetry, show that for any Hermitian A and B, the trace norm obeys

$$\|AB\|_{\rm tr} = \|BA\|_{\rm tr} . \qquad (2)$$

[**Hint**: Show that $BAAB$ and $ABBA$ have the same eigenvalues.]

The *overlap* of two probability distributions $\{p_i\}$ and $\{\tilde{p}_i\}$ is defined as

$${\rm Overlap}(\{p_i\},\{\tilde{p}_i\}) \equiv \sum_i \sqrt{p_i\cdot\tilde{p}_i} . \qquad (3)$$

Suppose that we try to distinguish the two states ρ and $\tilde{\rho}$ by performing the POVM $\{E_i\}$. Then the two corresponding probability distributions have the overlap

$${\rm Overlap}(\rho,\tilde{\rho};\{E_i\}) \equiv \sum_i \sqrt{{\rm tr}\ \rho E_i}\cdot\sqrt{{\rm tr}\ \tilde{\rho}E_i} . \qquad (4)$$

It turns out that the minimal overlap that can be achieved by any POVM is

$$\min_{\{E_i\}}\left[{\rm Overlap}(\rho,\tilde{\rho};\{E_i\})\right] = \|\tilde{\rho}^{\frac{1}{2}}\rho^{\frac{1}{2}}\|_{\rm tr} = \sqrt{F(\rho,\tilde{\rho})} . \qquad (5)$$

In this exercise, you will show that the square root of the fidelity bounds the overlap, but not that the bound can be saturated.

b) The space of linear operators acting on a Hilbert space is itself a Hilbert space, where the inner product (A, B) of two operators A and B is

$$(A, B) \equiv \mathrm{tr}\left(A^\dagger B\right) \ . \tag{6}$$

For this inner product, the Schwarz inequality becomes

$$|\mathrm{tr}\ A^\dagger B| \leq \left(\mathrm{tr}\ A^\dagger A\right)^{1/2} \left(\mathrm{tr}\ B^\dagger B\right)^{1/2} \ , \tag{7}$$

Choosing $A = \rho^{\frac{1}{2}} E_i^{\frac{1}{2}}$ and $B = U \tilde{\rho}^{\frac{1}{2}} E_i^{\frac{1}{2}}$ (for an arbitrary unitary U), use this form of the Schwarz inequality to show that

$$\mathrm{Overlap}(\rho, \tilde{\rho}; \{E_i\}) \geq |\mathrm{tr}\ \rho^{\frac{1}{2}} U \tilde{\rho}^{\frac{1}{2}}| \ . \tag{8}$$

c) Now use the polar decomposition

$$A = V\sqrt{A^\dagger A} \tag{9}$$

(where V is unitary) to write

$$\tilde{\rho}^{\frac{1}{2}} \rho^{\frac{1}{2}} = V \sqrt{\rho^{\frac{1}{2}} \tilde{\rho} \rho^{\frac{1}{2}}} \ , \tag{10}$$

and by choosing the unitary U in eq. (8) to be $U = V^{-1}$, show that

$$\mathrm{Overlap}(\rho, \tilde{\rho}; \{E_i\}) \geq \mathrm{tr}\ \sqrt{\rho^{\frac{1}{2}} \tilde{\rho} \rho^{\frac{1}{2}}} \ . \tag{11}$$

d) We can obtain an explicit formula for the fidelity in the case of two states of a single qubit. Using the Bloch parametrization

$$\rho(\vec{P}) = \frac{1}{2}\left[I + \vec{\sigma} \cdot \vec{P}\right] \ , \tag{12}$$

show that the fidelity of two single-qubit states with polarization vectors $\vec{P}$ and $\vec{Q}$ is

$$F(\vec{P}, \vec{Q}) = \frac{1}{2}\left(1 + \vec{P} \cdot \vec{Q} + \sqrt{(1 - \vec{P}^2)(1 - \vec{Q}^2)}\right) \ . \tag{13}$$

[**Hint**: First note that the eigenvalues of a 2×2 matrix can be expressed in terms of the trace and determinant of the matrix. Then evaluate the determinant and trace of $\left(\rho^{\frac{1}{2}} \tilde{\rho} \rho^{\frac{1}{2}}\right)$, and calculate the fidelity using the corresponding expression for the eigenvalues.]

2.2 Eavesdropping and disturbance

Alice wants to send a message to Bob. Alice is equipped to prepare either one of the two states $|u\rangle$ or $|v\rangle$. These two states, in a suitable basis, can be expressed as

$$|u\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} , \quad |v\rangle = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} , \tag{14}$$

where $0 < \alpha < \pi/4$. Suppose that Alice decides at random to send either $|u\rangle$ or $|v\rangle$ to Bob, and Bob is to make a measurement to determine what she sent. Since the two states are not orthogonal, Bob cannot distinguish the states perfectly.

a) Bob realizes that he can't expect to be able to identify Alice's qubit every time, so he settles for a procedure that is successful only some of the time. He performs a POVM with three possible outcomes: $\neg u$, $\neg v$, or DON'T KNOW. If he obtains the result $\neg u$, he is certain that $|v\rangle$ was send, and if he obtains $\neg v$, he is certain that $|u\rangle$ was sent. If the result is DON'T KNOW, then his measurement is inconclusive. This POVM is defined by the operators

$$\begin{aligned} \boldsymbol{F}_{\neg u} = A(\mathbf{1} - |u\rangle\langle u|) , \quad \boldsymbol{F}_{\neg v} = A(\mathbf{1} - |v\rangle\langle v|) , \\ \boldsymbol{F}_{\mathrm{DK}} = (1-2A)\mathbf{1} + A\left(|u\rangle\langle u| + |v\rangle\langle v|\right) , \end{aligned} \tag{15}$$

where A is a positive real number. How should Bob choose A to minimize the probability of the outcome DK, and what is this minimal DK probability (assuming that Alice chooses from $\{|u\rangle, |v\rangle\}$ equiprobably)? [**Hint:** If A is too large, $\boldsymbol{F}_{\mathrm{DK}}$ will have negative eigenvalues, and Eq.(15) will not be a POVM.]

b) Eve also wants to know what Alice is sending to Bob. Hoping that Alice and Bob won't notice, she intercepts each qubit that Alice sends, by performing an orthogonal measurement that projects onto the basis $\left\{\binom{1}{0}, \binom{0}{1}\right\}$. If she obtains the outcome $\binom{1}{0}$, she sends the state $|u\rangle$ on to Bob, and if she obtains the outcome $\binom{0}{1}$, she sends $|v\rangle$ on to Bob. Therefore each time Bob's POVM has a conclusive outcome, Eve knows with certainty what that outcome is. But Eve's tampering causes detectable errors; sometimes Bob obtains a "conclusive" outcome that actually differs from what Alice sent. What is the probability of such an error?

2.3 Approximate cloning

The *no-cloning* theorem shows that we can't build a unitary machine that will make a perfect copy of an unknown quantum state. But suppose we are willing to settle for an *imperfect* copy — what fidelity might we achieve?

Consider a machine that acts on three qubit states according to

$$\begin{aligned} |000\rangle_{ABC} &\rightarrow \sqrt{\frac{2}{3}}|00\rangle_{AB}|0\rangle_C + \sqrt{\frac{1}{3}}|\psi^+\rangle_{AB}|1\rangle_C \\ |100\rangle_{ABC} &\rightarrow \sqrt{\frac{2}{3}}|11\rangle_{AB}|1\rangle_C + \sqrt{\frac{1}{3}}|\psi^+\rangle_{AB}|0\rangle_C \ . \end{aligned} \qquad (16)$$

a) Is such a device physically realizable, in principle?

If the machine operates on the initial state $|\psi\rangle_A|00\rangle_{BC}$, it produces an pure entangled state $|\Psi\rangle_{ABC}$ of the three qubits. But if we observe qubit A alone, its final state is the density operator $\rho'_A = \mathrm{tr}_{BC}\left(|\Psi\rangle\langle\Psi|\right)$. Similarly, the qubit B, observed in isolation, has the final state ρ'_B. It is easy to see that $\rho'_A = \rho'_B$ — these are the identical, but imperfect, copies of the input pure state $|\psi\rangle_A$.

b) The mapping from the initial state $|\psi\rangle\langle\psi|$ to the final state ρ'_A of qubit A defines a quantum channel $\mathcal{E}$. Find an operator-sum representation of $\mathcal{E}$.

c) For $|\psi\rangle_A = a|0\rangle_A + b|1\rangle_A$, find ρ'_A, and compute its fidelity $F \equiv \langle\psi|\rho'_A|\psi\rangle$ with the input state $|\psi\rangle$.

2.4 Hardy's theorem

Bob (in Boston) and Claire (in Chicago) share many identically prepared copies of the two-qubit state

$$|\psi(x)\rangle = \sqrt{(1-2x)}\ |00\rangle + \sqrt{x}\ |01\rangle + \sqrt{x}\ |10\rangle\ , \qquad (17)$$

where x is a real number between 0 and 1/2. They conduct many trials in which each measures his/her qubit in the basis $\{|0\rangle, |1\rangle\}$, and they learn that if Bob's outcome is 1 then Claire's is always 0, and if Claire's outcome is 1 then Bob's is always 0.

Bob and Claire conduct further experiments in which Bob measures in the basis $\{|0\rangle, |1\rangle\}$ and Claire measures in the orthonormal basis $\{|\varphi\rangle, |\varphi^\perp\rangle\}$. They discover that if Bob's outcome is 0, then Claire's outcome is always φ and never $\varphi^\perp$. Similarly, if Claire measures in the basis $\{|0\rangle, |1\rangle\}$ and Bob measures in the basis $\{|\varphi\rangle, |\varphi^\perp\rangle\}$, then if Claire's outcome is 0, Bob's outcome is always φ and never $\varphi^\perp$.

a) Express the basis $\{|\varphi\rangle, |\varphi^\perp\rangle\}$ in terms of the basis $\{|0\rangle, |1\rangle\}$.

Bob and Claire now wonder what will happen if they both measure in the basis $\{|\varphi\rangle, |\varphi^\perp\rangle\}$. Their friend Albert, a firm believer in local hidden variables, predicts that it is impossible for both to obtain the outcome $\varphi^\perp$ (a prediction known as *Hardy's theorem*). Albert argues as follows:

> When both Bob and Claire measure in the basis $\{|\varphi\rangle, |\varphi^\perp\rangle\}$, it is reasonable to consider what might have happened if one or the other had measured in the basis $\{|0\rangle, |1\rangle\}$ instead.
>
> So suppose that Bob and Claire both measure in the basis $\{|\varphi\rangle, |\varphi^\perp\rangle\}$, and that they both obtain the outcome $\varphi^\perp$. Now if Bob had measured in the basis $\{|0\rangle, |1\rangle\}$ instead, we can be certain that his outcome would have been 1, since experiment has shown that if Bob had obtained 0 then Claire could not have obtained $\varphi^\perp$. Similarly, if Claire had measured in the basis $\{|0\rangle, |1\rangle\}$, then she certainly would have obtained the outcome 1. We conclude that if Bob and Claire both measured in the basis $\{|0\rangle, |1\rangle\}$, both would have obtained the outcome 1. But this is a contradiction, for experiment has shown that it is not possible for both Bob and Claire to obtain the outcome 1 if they both measure in the basis $\{|0\rangle, |1\rangle\}$.
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> We are therefore forced to conclude that if Bob and Claire both measure in the basis $\{|\varphi\rangle, |\varphi^\perp\rangle\}$, it is impossible for both to obtain the outcome $\varphi^\perp$.

Though impressed by Albert's reasoning, Bob and Claire decide to investigate what predictions can be inferred from quantum mechanics.

b) If Bob and Claire both measure in the basis $\{|\varphi\rangle, |\varphi^\perp\rangle\}$, what is the quantum-mechanical prediction for the probability $P(x)$ that both obtain the outcome $\varphi^\perp$?

c) Find the "maximal violation" of Hardy's theorem: show that the maximal value of $P(x)$ is $P[(3-\sqrt{5})/2] = (5\sqrt{5}-11)/2 \approx .0902$.

d) Bob and Claire conduct an experiment that confirms the prediction of quantum mechanics. What was wrong with Albert's reasoning?