

# Ph 219a/CS 219a

## Exercises

Due: Wednesday 26 November 2008

### 3.1 Hardy's theorem

Bob (in Boston) and Claire (in Chicago) share many identically prepared copies of the two-qubit pure state

$$|\psi(x)\rangle = \sqrt{(1-2x)} |00\rangle + \sqrt{x} |01\rangle + \sqrt{x} |10\rangle, \quad (1)$$

where  $x$  is a real number between 0 and 1/2. They conduct many trials in which each measures his/her qubit in the basis  $\{|0\rangle, |1\rangle\}$ , and they learn that if Bob's outcome is 1 then Claire's is always 0, and if Claire's outcome is 1 then Bob's is always 0.

Bob and Claire conduct further experiments in which Bob measures in the basis  $\{|0\rangle, |1\rangle\}$  and Claire measures in the orthonormal basis  $\{|\varphi\rangle, |\varphi^\perp\rangle\}$ . They discover that if Bob's outcome is 0, then Claire's outcome is always  $\varphi$  and never  $\varphi^\perp$ . Similarly, if Claire measures in the basis  $\{|0\rangle, |1\rangle\}$  and Bob measures in the basis  $\{|\varphi\rangle, |\varphi^\perp\rangle\}$ , then if Claire's outcome is 0, Bob's outcome is always  $\varphi$  and never  $\varphi^\perp$ .

a) Express the basis  $\{|\varphi\rangle, |\varphi^\perp\rangle\}$  in terms of the basis  $\{|0\rangle, |1\rangle\}$ .

Bob and Claire now wonder what will happen if they both measure in the basis  $\{|\varphi\rangle, |\varphi^\perp\rangle\}$ . Their friend Albert, a firm believer in local realism, predicts that it is impossible for both to obtain the outcome  $\varphi^\perp$  (a prediction known as *Hardy's theorem*). Albert argues as follows:

When both Bob and Claire measure in the basis  $\{|\varphi\rangle, |\varphi^\perp\rangle\}$ , it is reasonable to consider what might have happened if one or the other had measured in the basis  $\{|0\rangle, |1\rangle\}$  instead.

So suppose that Bob and Claire both measure in the basis  $\{|\varphi\rangle, |\varphi^\perp\rangle\}$ , and that they both obtain the outcome  $\varphi^\perp$ . Now if Bob had measured in the basis  $\{|0\rangle, |1\rangle\}$  instead, we can be certain that his outcome would have been 1, since experiment has shown that if Bob had obtained 0 then Claire could

not have obtained  $\varphi^\perp$ . Similarly, if Claire had measured in the basis  $\{|0\rangle, |1\rangle\}$ , then she certainly would have obtained the outcome 1. We conclude that if Bob and Claire both measured in the basis  $\{|0\rangle, |1\rangle\}$ , both would have obtained the outcome 1. But this is a contradiction, for experiment has shown that it is not possible for both Bob and Claire to obtain the outcome 1 if they both measure in the basis  $\{|0\rangle, |1\rangle\}$ .

We are therefore forced to conclude that if Bob and Claire both measure in the basis  $\{|\varphi\rangle, |\varphi^\perp\rangle\}$ , it is impossible for both to obtain the outcome  $\varphi^\perp$ .

Though impressed by Albert's reasoning, Bob and Claire decide to investigate what predictions can be inferred from quantum mechanics.

- b) If Bob and Claire both measure in the basis  $\{|\varphi\rangle, |\varphi^\perp\rangle\}$ , what is the quantum-mechanical prediction for the probability  $P(x)$  that both obtain the outcome  $\varphi^\perp$ ?
- c) Find the "maximal violation" of Hardy's theorem: show that the maximal value of  $P(x)$  is  $P[(3 - \sqrt{5})/2] = (5\sqrt{5} - 11)/2 \approx .0902$ .
- d) Bob and Claire conduct an experiment that confirms the prediction of quantum mechanics. What was wrong with Albert's reasoning?

### 3.2 Closing the detection loophole

Recall that the *CHSH inequality*

$$|\langle ab \rangle + \langle a'b \rangle + \langle ab' \rangle - \langle a'b' \rangle| \leq 2 \quad (2)$$

holds if the random variables  $a, b, a', b'$  take values  $\pm 1$  and are governed by a joint probability distribution. The maximal violation of this inequality by the quantum-mechanical predictions occurs when the left-hand-side is  $2\sqrt{2}$ , which is achieved if Alice and Bob share the maximally entangled state  $|\phi^+\rangle$ ,  $a, a'$  are measurements of Alice's qubit along axes  $\hat{x}$  and  $\hat{z}$ , and  $b, b'$  are measurements of Bob's qubit along axes  $(\hat{x} + \hat{z})/\sqrt{2}$  and  $(\hat{x} - \hat{z})/\sqrt{2}$ .

Alice and Bob have done a beautiful experiment measuring the polarizations of entangled photon pairs, and have confirmed the CHSH inequality violation predicted by quantum mechanics. Albert is skeptical. He points out that the detectors used by Alice and Bob in their

experiment are not very efficient. Usually, when Alice detects a photon, Bob does not, and when Bob detects a photon, Alice does not. Therefore, they discard the data for most of the photon pairs, and retain the results only in the case when two photons are detected in coincidence. In their analysis of the data, Alice and Bob assume that their results are based on a fair sample of the probability distribution governing the measured variables. But Albert argues that their conclusions could be evaded if *whether* a photon is detected is correlated with the *outcome* of the polarization measurement.

Alice and Bob wonder how much they will need to improve their detector efficiency to do an experiment that will impress Albert.

Alice can choose to orient her detector along any axis. Ideally, when she aligns the detector with the axis  $\hat{a}$ , the detector “clicks” with probability one if the qubit points up along the axis  $\hat{a}$ , and it clicks with probability zero if the qubit points down along  $\hat{a}$ . But the detector is imperfect — though the detector never clicks when the qubit points down, it clicks with probability  $\eta_A < 1$  when the qubit points up; we say that  $\eta_A$  is the detector’s *efficiency*. Thus, when the detector clicks, Alice is sure the qubit is up, but when it fails to click, she can’t be sure whether the qubit is up or down. Bob has a similar detector with efficiency  $\eta_B$ .

Consider a local hidden variable model intended to describe Alice’s and Bob’s observations. Alice can make either one of two measurements, labeled  $a$  and  $a'$ , and Bob can make either one of two measurements, labeled  $b$  and  $b'$ . When the hidden variables are in a definite *local configuration*, it is determined whether Alice’s detector will click if Alice measures  $a$  and whether it will click if Alice measures  $a'$ ; similarly, it is determined whether Bob’s detector will click if Bob measures  $b$  and whether it will click if Bob measures  $b'$ . Define a variable  $x \in \{0, 1\}$  indicating whether Alice’s detector will click if she measures  $a$ :  $x = 1$  if it will click and  $x = 0$  if not. Similarly,  $x' \in \{0, 1\}$  indicates whether Alice’s detector will click if she measures  $a'$ ,  $y \in \{0, 1\}$  indicates whether Bob’s detector will click if he measures  $b$  and  $y' \in \{0, 1\}$  indicates whether Bob’s detector will click if he measures  $b'$ .

Alice and Bob are free to decide how to align their detectors in each measurement; therefore, after many trials, they will be able to estimate with high statistical confidence the probabilities  $P_{++}(ab)$ ,  $P_{++}(a'b)$ ,

$P_{++}(ab')$ , and  $P_{++}(a'b')$ . Here *e.g.*  $P_{++}(ab)$  denotes the probability that both detectors click if Alice measures  $a$  and Bob measures  $b$  (including the effects of detector inefficiency).

a) If  $x, x', y, y' \in \{0, 1\}$ , show that

$$xy + xy' + x'y - x'y' \leq x + y . \quad (3)$$

b) Show that, in a local hidden variable model (*i.e.*, for any probability distribution on local configurations),

$$P_{++}(ab) + P_{++}(a'b) + P_{++}(ab') - P_{++}(a'b') \leq P_{+}(a) + P_{+}(b) ; \quad (4)$$

here  $P_{+}(a)$  denotes the probability that Alice's detector clicks if oriented along  $a$ , and  $P_{+}(b)$  denotes the probability that Bob's detector clicks if oriented along  $b$ .

c) Now compare with the predictions of quantum mechanics, where Alice's detector has efficiency  $\eta_A$  and Bob's detector has efficiency  $\eta_B$ . Choosing the  $a, a', b, b'$  that maximally violate the CHSH inequality, show that the quantum-mechanical predictions violate eq. (4) only if

$$\frac{\eta_A \eta_B}{\eta_A + \eta_B} > \frac{1}{1 + \sqrt{2}} . \quad (5)$$

Thus, if  $\eta_A = \eta_B$ , Alice and Bob require detectors with efficiency above 82.84% to overcome Albert's objection.

### 3.3 We're so sorry, Uncle Albert

Consider the  $n$ -qubit "cat" state

$$|\psi_n\rangle = \sqrt{\frac{1}{2}}(|000\dots 0\rangle + |111\dots 1\rangle) . \quad (6)$$

a) Show that  $|\psi_n\rangle$  is an eigenstate of the Hermitian operator

$$\Sigma = (\mathbf{X} + i\mathbf{Y})^{\otimes n} + (\mathbf{X} - i\mathbf{Y})^{\otimes n} , \quad (7)$$

and compute its eigenvalue; here  $\mathbf{X} = \sigma_1$  and  $\mathbf{Y} = \sigma_2$  are Pauli operators.

- b) Consider a local hidden variable theory for  $n$ , where the  $j$ th party can measure either  $X_j$  or  $Y_j$ . In each *local configuration* and for each  $j$ ,  $X_j$  and  $Y_j$  take definite values in the set  $\{1, -1\}$ . Find the possible values of

$$\Sigma = \prod_{j=1}^n (X_j + iY_j) + \prod_{j=1}^n (X_j - iY_j) \quad (8)$$

that can occur in a local configuration.

- c) Find an upper satisfied by  $\langle \Sigma \rangle$  for any convex combination of local configurations. Compare with (a). What would Einstein say?

### 3.4 Coherent classical communication

We saw that the tasks realized by superdense coding (SD) and by teleportation (TP) can be succinctly expressed as *resource inequalities*:

$$\begin{aligned} [q \rightarrow q] + [qq] &\geq 2[c \rightarrow c] && \text{(SD) ,} \\ 2[c \rightarrow c] + [qq] &\geq [q \rightarrow q] && \text{(TP) .} \end{aligned} \quad (9)$$

Here  $[c \rightarrow c]$  denotes one classical bit (cbit) sent from Alice to Bob,  $[q \rightarrow q]$  denotes one qubit sent from Alice to Bob, and  $[qq]$  denotes one *e-bit* — a maximally entangled pair of qubits shared by Alice and Bob. The meaning of the inequality is that, using only local operations, the input resources on the left can be converted into the output resources on the right.

These inequalities are strict, in the sense that the resource conversions are irreversible — there is no protocol corresponding to SD or TP “running backwards.” It turns out that there is a natural way to formulate versions of SD and TP that are reversible in the sense that the resource inequality can be replaced by an equality, but to do so we must replace classical communication with a stronger resource: *coherent classical communication*.

The three types of communication that we wish to consider can be realized as isometries (inner-product preserving linear maps). Sending a qubit from Alice to Bob can be expressed as

$$[q \rightarrow q] : |x\rangle_A \rightarrow |x\rangle_B , \quad (10)$$

where  $x \in \{0, 1\}$ . Sending a cbit from Alice to Bob can be expressed as

$$[c \rightarrow c] : |x\rangle_A \rightarrow |x\rangle_B \otimes |x\rangle_E ; \quad (11)$$

here  $E$  denotes an environment that cannot be accessed by either Alice or Bob. The communication is classical because a quantum signal that Alice attempts to send to Bob decoheres in the basis  $\{|0\rangle, |1\rangle\}$ . Sending a *cobit* (a unit of coherent classical communication) from Alice to Bob can be expressed as

$$[q \rightarrow qq] : |x\rangle_A \rightarrow |x\rangle_A \otimes |x\rangle_B ; \quad (12)$$

this is somewhat like classical communication, except that Alice maintains control of the “environment.”

a) Show that

$$[q \rightarrow q] \geq [q \rightarrow qq] \geq [c \rightarrow c] , \quad \text{and} \quad [q \rightarrow qq] \geq [qq] . \quad (13)$$

(That is, explain how to use what is on the left side of each inequality, together with local operations, to achieve what is on the right side.)

b) Show that

$$[q \rightarrow q] + [qq] \geq 2[q \rightarrow qq] . \quad (14)$$

**Hint:** Use a coherent version of superdense coding, where Bob performs a local unitary transformation  $U_B$  rather than a Bell measurement. That is, Alice has two qubits in the state  $|xy\rangle_A$  and she shares a Bell pair with Bob. To achieve  $|xy\rangle_A \rightarrow |xy\rangle_A \otimes |xy\rangle_B$ , first Alice applies a local unitary transformation  $U_A$  to her two qubits and her half of the entangled pair, then she sends her half of the Bell pair to Bob, and finally Bob applies a unitary transformation  $U_B$ . The problem is to find  $U_A$  and  $U_B$ .

c) Show that

$$2[q \rightarrow qq] + [qq] \geq [q \rightarrow q] + 2[qq] . \quad (15)$$

**Hint:** Use a coherent version of teleportation, where Alice performs a local unitary transformation  $V_A$  rather than a Bell measurement. That is, Alice applies  $V_A$ , then she sends two cobits to Bob, and finally Bob applies a local unitary transformation  $V_B$ . The problem is to find  $V_A$  and  $V_B$ . Be sure to verify that Alice and Bob wind up with 2 ebits at the end of the protocol.

**Remark:** We see that coherent teleportation actually creates more entanglement than it consumes. We can express the results of (b) and (c) by saying

$$2[q \rightarrow qq] = [q \rightarrow q] + [qq] , \quad (16)$$

but where the equality indicates that the resource conversion could be *catalytic* — a resource might need to be borrowed to activate the process, but this borrowed resource can be returned when the protocol is completed.