Ph 219/CS 219

Exercises
Due: Friday 10 March 2006

7.1 Finding a collision

Suppose that a black box evaluates a function

$$f : \{0,1\}^n \to \{0,1\}^{n-1} \ . \tag{1}$$

We are promised that the function is 2-to-1, and we are to find a "collision" – values x and y such that $f(x) = f(y)$. This problem is harder than Simon's problem, because we are not promised that the function is periodic. Let $N = 2^n$.

a) Describe a randomized classical algorithm that requires SPACE = $O(\sqrt{N})$ and that succeeds in finding a collision with high probability in $O(\sqrt{N})$ queries of the black box.

b) Now suppose that only SPACE = $O(N^{1/3})$ is available. Describe a randomized classical algorithm that finds a collision with high probability in $O(N^{2/3})$ queries.

c) Show that Grover's exhaustive search algorithm can be used to find a collision in $O(\sqrt{N})$ quantum queries, using SPACE = $O(1)$.

d) Describe a quantum algorithm that uses SPACE = $O(M)$ and finds a collision in $O(M) + O(\sqrt{N/M})$ quantum queries. [**Hint**: First query the box M times to learn the value of $f(x)$ for M arguments $\{x_1, x_2, \ldots, x_M\}$, then search for y such that $f(y) = f(x_i)$ for some x_i.] Thus, if M is chosen to optimize the number of queries, the quantum algorithm uses SPACE = $O(N^{1/3})$ and $O(N^{1/3})$ quantum queries.

7.2 All the information for half the price

A black box computes a function

$$f : \{0,1\}^n \to \{0,1\} \ . \tag{2}$$

This function can be represented by a binary string

$$X = X_{N-1} X_{N-2} \cdots X_1 X_0 \ , \tag{3}$$

where $X_i = f(i)$ and $N = 2^n$. Our goal is to obtain, with high probability of success, *complete* information about the box; that is, to find the value of X. The only resource we care about is the number of queries of the box — TIME and SPACE are otherwise unlimited.

a) How many classical queries are needed to find X with success probability at least $2/3$?

b) Suppose that the state

$$|\Psi_{X,N}\rangle = \frac{1}{\sqrt{2^N}} \sum_{Y\in\{0,1\}^N} (-1)^{X\cdot Y}|Y\rangle \tag{4}$$

has been prepared, where the sum is over all N-bit strings, and $X \cdot Y$ denotes the mod 2 bitwise inner product

$$\begin{aligned} X \cdot Y \quad &= \quad (X_{N-1} \cdot Y_{N-1}) \oplus (X_{N-2} \cdot Y_{N-2}) \\ &\qquad \cdots \oplus (X_1 \cdot Y_1) \oplus (X_0 \cdot Y_0) \ . \end{aligned} \tag{5}$$

Describe a way, by applying a simple unitary and then a measurement, to find the value of X with certainty.

c) Explain how the unitary transformation

$$U : |Y\rangle \to (-1)^{X\cdot Y}|Y\rangle \tag{6}$$

can be implemented with $|Y|$ queries of the box, where $|Y|$ denotes the *Hamming weight* of Y, the number of 1's in the string.

d) Suppose we prepare the state

$$|\Phi_K\rangle = \frac{1}{\sqrt{M_K}} \sum_{Y:|Y|\le K} |Y\rangle \ , \tag{7}$$

where

$$M_K = \sum_{j=0}^{K} \binom{N}{j} \ , \tag{8}$$

and then apply U (requiring at most K queries) to obtain

$$|\Psi_{X,K}\rangle = \frac{1}{\sqrt{M_K}} \sum_{Y:|Y|\le K} (-1)^{X\cdot Y}|Y\rangle \ , \tag{9}$$

Show that, by applying the procedure that you described in your answer to (*b*), we can determine the value of X with a probability of success

$$P_{\rm succ}(N,K) = |\langle\Psi_{X,K}|\Psi_{X,N}\rangle|^2 \ , \tag{10}$$

and compute the value of $P_{\rm succ}(N,K)$.

e) Suppose that

$$K = N/2 + c\sqrt{N} \ , \tag{11}$$

where c is a constant. Show that

$$1 - P_{\rm succ}(N,K) = O(e^{-2c^2}) \ . \tag{12}$$

Thus we can extract all the information from the box in a number of queries $(N/2)\cdot[1+O(1/\sqrt{N})]$.

7.3 Quantum counting

A black box computes a function

$$f : \{0,1\}^n \to \{0,1\} \ , \tag{13}$$

which can be represented by a binary string

$$X = X_{N-1}X_{N-2}\cdots X_1X_0 \ , \tag{14}$$

where $X_i = f(i)$ and $N = 2^n$. Our goal is to count the number r of states "marked" by the box; that is, to determine the Hamming weight $r = |X|$ of X. We can devise a quantum algorithm that counts the marked states by combining Grover's exhaustive search with the quantum Fourier transform.

a) Suppose we can consult a quantum oracle that executes the unitary transformation U. We'd like to perform $\Lambda(U)$, the unitary U conditioned on the value of a control qubit. Devise a quantum circuit with one oracle query that executes $\Lambda(U)$, using ancilla qubits and Λ(SWAP) gates, where

$$\text{SWAP} : |x\rangle|y\rangle \to |y\rangle|x\rangle \ . \tag{15}$$

b) Let

$$|\Psi_X\rangle = \frac{1}{\sqrt{r}} \sum_{j:X_j=1} |j\rangle \tag{16}$$

denote the uniform superposition of the marked states, and let $U_{\rm Grover}$ denote the "Grover iteration," which performs a rotation by the angle 2θ in the plane spanned by $|\Psi_X\rangle$ and

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N} |j\rangle \ , \tag{17}$$

where

$$\sin\theta = \langle s|\Psi_X\rangle = \sqrt{\frac{r}{N}}\ . \tag{18}$$

Consider a unitary transformation

$$V : |t\rangle \otimes |\Phi\rangle \to |t\rangle \otimes U^t_{\text{Grover}}|\Phi\rangle \tag{19}$$

that reads a counter register taking values $t \in \{0, 1, 2, \ldots, T-1\}$ (where $T = 2^m$), and then applies U_{Grover} t times. Explain how V can be implemented, calling the oracle $T-1$ times. [**Hint:** Use the binary expansion $t = \sum_{k=0}^{m-1} t_k 2^k$ and the conditional oracle call from (a).]

c) Suppose that $r \ll N$. Show that, by applying V, performing the quantum Fourier transform on the counter register, and then measuring the counter register, we can determine θ to accuracy $O(1/T)$, and hence we can find r with high success probability in $T = O(\sqrt{rN})$ queries. Compare to the best classical protocol.