Ph/CS 219, Winter 2007 **Quantum Computation** **Prof. Alexei Kitaev**	**Problem set #3** **Due Tuesday, March 13, 2007**

Anyons based on a finite group

This whole problem set is concerned with S_3 anyons. Here is a brief summary of the model and its basic properties. Please use it as a notation reference. It may look confusing at first but, hopefully, Problems 3.1 and 3.2 will help you to get used to the formalism.

The lattice model

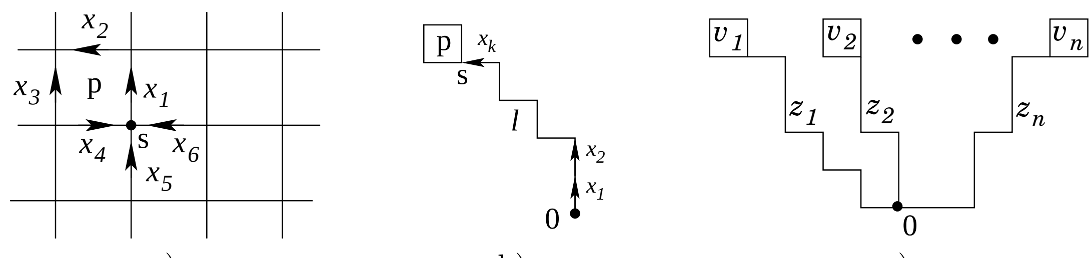


Figure 1: a) Qudits surrounding a site and a plaquette; b) path l used in the definition of $z = x_k \cdots x_2 x_1$; c) a set of n vortices (to illustrate the assignment of the variables v_j, z_j).

On each edge of the square lattice, there is a qudit with $|G|$ basis states indexed by a group element $x \in G$. (In the specific questions below, G is assumed to be the permutation group on 3 elements, denoted by S_3.) The labeling of the basis vectors depends on the orientation of the edge; if the orientation changes, one needs to replace $|x\rangle$ by $|x^{-1}\rangle$. For each site s and each group element g we define an operator A_s^g that flips the qudits on incident edges: $x_j \mapsto gx_j$ if edge j is oriented inward or $x_j \mapsto x_j g^{-1}$ if it is oriented outward. (In the example shown in Fig. 1a, $A^g|x_1, x_4, x_5, x_6\rangle = |gx_1, x_4 g^{-1}, gx_5, gx_6\rangle$, the other qudits being unchanged.) For each plaquette p and site s on its boundary, $B_{p,s}^h$ denotes the projector onto the subspace spanned by basis states $|\vec{x}\rangle = |x_1, \ldots, x_N\rangle$ such that the product of x's around p starting at s (e.g., $v = x_4 x_3^{-1} x_2 x_1$ in Fig. 1a) is equal to h^{-1}. The Hamiltonian of the model was defined in class. The ground state (also called "vacuum") satisfies these stabilizer conditions:

$$A_s^g|\mathrm{vac}\rangle = |\mathrm{vac}\rangle, \qquad B_{p,s}^h|\mathrm{vac}\rangle = \delta_{h,1}|\mathrm{vac}\rangle \qquad \text{for all } s, p. \tag{1}$$

(Here $\delta_{h,1}$ is the Kronecker symbol: $\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0$ if $x \neq y$.) The operators A^g, B^h (associated with given s and p) obey the following relations:

$$A^{g_1} A^{g_2} = A^{g_1 g_2}, \qquad B^{h_1} B^{h_2} = \delta_{h_1,h_2} B^{h_1}, \qquad A^g B^h = B^{ghg^{-1}} A^g. \tag{2}$$

The operator algebra generated by operators with these relations is called the *Drinfeld double* of G. (It actually has a lot of additional structure that we do not consider.)

Quasiparticles may classified as *charges* (located on sites) and *pure vortices* (located on plaquettes). A more general type of excitation occupies a plaquette p and a site s on its boundary. We call such excitation *dyons*, though some of them are similar to vortices from the symmetry point of view (see Problem 3.2). A dyon at the given location is characterized by a pair of group elements, z and v, where z is the product of edge labels along some path l from the origin (e.g., $z = x_k \cdots x_2 x_1$ in Fig. 1b), and v is the *local flux*, i.e., the label product around p. For our purposes, it is convenient to use a *topological flux* $w = z^{-1}vz$ instead of v; the topological flux corresponds to the path that follows l, goes around the vortex counterclockwise, and traverses l backward.

Thus, a general excitation is described by z and w. Charges have $w = 1$, so the only relevant variable is z. For pure vortices, z forms the fully symmetric state $|G|^{-1/2}\sum_{z\in G}|z\rangle$ (which can be factored out), so the relevant variable is w.

Each quasiparticle is characterized by "flavor" and "color". Intuitively, the flavor is a local degree of freedom (like the spin of an electron), which is susceptible to perturbations. The part that is not accessible locally is called "color". It is more elusive but yet more important because one can use it for quantum coding. Identifying flavor and color variables is a nontrivial problem. Rather, we first describe *operators* that act on flavor and color. The flavor operators include $A^g_{\mathrm{fl}} = A^g_s$ and $B^h_{\mathrm{fl}} = B^h_{s,p}$ defined above; the first takes z to gz, and the second projects onto a certain value of local flux v, namely, $v = h^{-1}$. The color operators are ones that *commute with flavor*. Both types of operators form the same kind of algebra though they act differently:

$$\text{Flavor operators:} \qquad A^g_{\mathrm{fl}}|z,w\rangle = |gz,w\rangle, \qquad B^h_{\mathrm{fl}}|z,w\rangle = \delta_{h^{-1},zwz^{-1}}|z,w\rangle, \tag{3}$$

$$\text{Color operators:} \qquad A^g_{\mathrm{col}}|z,w\rangle = |zg^{-1},gwg^{-1}\rangle, \qquad B^h_{\mathrm{col}}|z,w\rangle = \delta_{h,w}|z,w\rangle. \tag{4}$$

If we forget about the action on z, then $v = zwz^{-1}$ is the flavor variable, and w is the color variable. However, z takes part in both flavor and color. Dividing it in two pieces requires the use of representation theory.

In the case of charges, we only have the variable z and the action of A^g on flavor and color:

$$A^g_{\mathrm{fl}}|z\rangle = |gz\rangle, \qquad A^g_{\mathrm{col}}|z\rangle = |zg^{-1}\rangle \tag{5}$$

(B^h acts trivially). Thus, a charge located at a single lattice site is described by the regular representation of G (actually, $G \times G$, because we have two actions of G). It splits as follows:

$$\mathcal{H} = \bigoplus_{\alpha} \underbrace{\mathcal{L}_{\bar{\alpha}}}_{\text{flavor}} \otimes \underbrace{\mathcal{L}_{\alpha}}_{\text{color}}. \tag{6}$$

In general, the color and flavor correspond to conjugate irreducible representations ($\bar{\alpha}$ and α) of the Drinfeld double.

Each irrep (i.e., particle type) is characterized by these data:

1. The flux is given by a conjugacy class $C \subseteq G$;

2. The charge of the quasiparticle is described by an irreducible representation of the centralizer $Z(u) = \{v \in G : uv = vu\}$, where $u \in C$ is arbitrary. (All elements in the same class have isomorphic centralizers.)

A particularly simple case is when the charge is trivial. Then the representation space has a standard basis $\{|u\rangle : u \in C\}$, and the generators of the Drinfeld double act as follows:

$$A^g|u\rangle = |gug^{-1}\rangle, \qquad B^h|u\rangle = \delta_{h,u}|u\rangle. \tag{7}$$

More examples are given below.

A single quasiparticle cannot be created from the vacuum. One can only create a set of quasiparticles that satisfies some global neutrality conditions. If the neutrality is not taken to account, the Hilbert space of n dyons is spanned by basis vectors $|z_1, w_1; \ldots; z_n, w_n\rangle$. Within this "raw" Hilbert space lies a *physical subspace* $\mathcal{L}$. Vectors in this subspace satisfy the condition $w_1 \ldots w_n = 1$ (i.e., the total flux is trivial) and are invariant under the action of the operator A^g at the origin (i.e., there is no extra charge there). More formally, $|\psi\rangle \in \mathcal{L}$ if and only if

$$A^g_{\text{global}}|\psi\rangle = |\psi\rangle, \qquad B^h_{\text{global}}|\psi\rangle = \delta_{h,1}|\psi\rangle, \tag{8}$$

where

$$\begin{aligned} A^g_{\text{global}}|z_1, w_1; \ldots; z_n, w_n\rangle &= |z_1 g^{-1}, g w_1 g^{-1}; \ldots; z_n g^{-1}, g w_n g^{-1}\rangle, \\ B^h_{\text{global}}|z_1, w_1; \ldots; z_n, w_n\rangle &= \delta_{h, w_1 \cdots w_n}|z_1, w_1; \ldots; z_n, w_n\rangle. \end{aligned} \tag{9}$$

(Note that these equations are only applicable to the set of *all* particles; they are not necessarily true if there is an extra charge at the origin or other vortices somewhere in the system.) The neutrality conditions mean that the particles form a color singlet. Indeed, A^g_{global}, B^h_{global} act on the particle colors.

Equation (9) is an example of a simultaneous action of an operator on several disjoint systems. We are used to the situation where the simultaneous action (on two systems) is described by $U \otimes U$ — the same unitary U is applied to both parts. The operator A^g_{global} is like that, but B^h_{global} is different. If we denote the simultaneous action of X on two systems by $\Delta(X)$, then

$$\Delta(A^g) = A^g \otimes A^g, \qquad \Delta(B^h) = \sum_{h_1 h_2 = h} B_{h_1} \otimes B_{h_2}. \tag{10}$$

The operation Δ is called *comultiplication*. It allows one to define the product of two representation. (For the product space to be a representation, we need some action of A^g, B^h on it, which is given by $\Delta(A^g)$, $\Delta(B^h)$.)

Representation of the group S_3 and its Drinfeld double

The group S_3 consists of permutation on 3 elements. The trivial permutation is denoted by e. Other permutations are denoted like this: (123) stands for the permutation $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$. There are three conjugacy classes, $C_1 = \{e\}$, $C_2 = \{(12), (13), (23)\}$, $C_3 = \{(123), (132)\}$ and three irreducible representations, $[+]$, $[-]$, $[2]$. All three irreps are self-conjugate, i.e., $\bar{\alpha} = \alpha$. The basis vectors in the representation $[2]$ are denoted by $|2_+\rangle$ and $|2_-\rangle$. The group acts on them as follows:

$$\mu|2_+\rangle = \omega|2_+\rangle, \quad \mu|2_-\rangle = \bar{\omega}|2_-\rangle, \qquad \sigma|2_+\rangle = |2_-\rangle, \quad \sigma|2_-\rangle = |2_+\rangle, \tag{11}$$

where $\mu = (123)$, $\sigma = (23)$, and $\omega = \frac{-1+i\sqrt{3}}{2}$.

Flux (conjugacy class)	Centralizer	Charge	Basis vectors
$C_1 = \{e\}$	$Z(e) = S_3$	$[+]$	$\|+\rangle$
		$[-]$	$\|-\rangle$
		$[2]$	$\|2_+\rangle,\ \|2_-\rangle$
$C_2 = \{(12),(23),(31)\}$	$Z((12)) = \{e,(12)\}$	$+$	$\|(12)\rangle,\ \|(23)\rangle,\ \|(31)\rangle$
		$-$	$\|(12),-\rangle,\ \|(23),-\rangle,\ \|(31),-\rangle$
$C_3 = \{(123),(132)\}$	$Z((123)) = \{e,(123),(132)\}$	1	$\|(123)\rangle,\ \|(132)\rangle$
		ω	$\|(123),\omega\rangle,\ \|(132),\omega\rangle$
		$\bar{\omega}$	$\|(123),\bar{\omega}\rangle,\ \|(132),\bar{\omega}\rangle$

Table 1: Types of S_3 anyons and notation for basis vectors in the corresponding representations.

Example. Let the group S_3 act on a space with basis vectors $|1\rangle$, $|2\rangle$, $|3\rangle$ by permutations, e.g.,

$$(123)|1\rangle = |2\rangle, \quad (123)|2\rangle = |3\rangle, \quad (123)|3\rangle = |1\rangle, \quad (23)|1\rangle = |1\rangle, \text{ etc.}$$

This representation splits into two irreps: $[+]$ and $[2]$. We can identify the corresponding basis vectors $|+\rangle$, $|2_+\rangle$, $|2_-\rangle$ with certain superpositions of $|1\rangle$, $|2\rangle$, $|3\rangle$:

$$\begin{aligned} |+\rangle &\mapsto \tfrac{1}{\sqrt{3}}\big(|1\rangle + |2\rangle + |3\rangle\big), \\ |2_+\rangle &\mapsto \tfrac{1}{\sqrt{3}}\big(|1\rangle + \bar{\omega}|2\rangle + \omega|3\rangle\big), \\ |2_-\rangle &\mapsto \tfrac{1}{\sqrt{3}}\big(|1\rangle + \omega|2\rangle + \bar{\omega}|3\rangle\big). \end{aligned}$$

Note that the actions of permutations on the left-hand side and the right-hand side (using the relevant definitions) are identical.

The fusion rules for the irreps are as follows:

$$[-]\times[-] = [+], \qquad [2]\times[-] = [2], \qquad [2]\times[2] = [+]+[-]+[2]. \tag{12}$$

(Of course, $[+]\times\alpha = \alpha$ or all α.) Let us give an explicit expression for the three irreps within $[2]\times[2]$:

$$\begin{gathered} |+\rangle \mapsto \tfrac{1}{\sqrt{2}}\big(|2_+,2_-\rangle + |2_-,2_+\rangle\big), \\ |-\rangle \mapsto \tfrac{1}{\sqrt{2}}\big(|2_+,2_-\rangle - |2_-,2_+\rangle\big), \\ |2_+\rangle \mapsto |2_-,2_-\rangle, \qquad |2_-\rangle \mapsto |2_+,2_+\rangle. \end{gathered} \tag{13}$$

Irreducible representations of the Drinfeld double are listed in Table 1. Let us describe the action of the operators A^g, B^h on the basis vectors from that table. In the case of trivial flux, A^g is the already familiar action of g, e.g., $A^{(123)}|2_+\rangle = \omega|2_+\rangle$, whereas B^h acts trivially: $B^h|\psi\rangle = \delta_{h,1}|\psi\rangle$. In the case of vortices with trivial charge, the action is described by Eq. (7), e.g., $A^{(12)}|(123)\rangle = |(132)\rangle$, $B^{(123)}|(132)\rangle = 0$. Finally, let us consider a dyon, e.g., the one given by flux C_3 and charge ω. As far as the action of A^g is concerned, the basis vectors $|(123),\omega\rangle$

and $|(132), \omega\rangle$ are transformed as $|2_+\rangle$ and $|2_-\rangle$:

$$\begin{aligned} A^{(123)}|(123),\omega\rangle &= \omega|(123),\omega\rangle, & A^{(23)}|(123),\omega\rangle &= |(132),\omega\rangle, \\ A^{(123)}|(132),\omega\rangle &= \bar{\omega}|(132),\omega\rangle, & A^{(23)}|(132),\omega\rangle &= |(123),\omega\rangle. \end{aligned} \tag{14}$$

Note that for both elements in the given conjugacy class, $u \in \{(123), (132)\}$, we have $A^u|u,\omega\rangle = \omega|u,\omega\rangle$. The operators B^h act on the dyon the same way as on a vortex:

$$B^h|u,\omega\rangle = \delta_{h,u}|u,\omega\rangle. \tag{15}$$

Problems

3.1 Fusion of S_3 charges. In this problem, we consider abstract charges described by representations of S_3. Four charges corresponding to the representation [2] can form a singlet in 3 distinct ways. Indeed, the first pair can fuse into [+], [−], or [2]. If it fuses into a representation α, the second pair must fuse into $\bar{\alpha}$ so that all four particles could form a singlet. Write each of these states, $|\xi_{+,+}\rangle$, $|\xi_{-,-}\rangle$, $|\xi_{2,2}\rangle$ explicitly in terms of $|2_+\rangle$, $|2_-\rangle$. **Hint:** Use Eq. (13).

3.2 Flavor and color. Consider a quasiparticle that violates the stabilizer conditions on some plaquette and some site on its boundary. As discussed above, flavor operators act on the quasiparticle itself whereas color operators act on some nonlocal degree of freedom. If we describe the quasiparticle by the variables z, w (which are themselves nonlocal), the flavor and color operators are given by Eqs. (3), (4). According to the general theory, the Hilbert space of the quasiparticle can be represented in the form (6), where α runs over irreducible representations of the Drinfeld double. We do not need a lot of abstract representation theory here. Rather, we will write the terms of this decomposition explicitly, in the form

$$|\text{flavor}\rangle \otimes |\text{color}\rangle \mapsto \sum_{z,w} c_{z,w}|z,w\rangle.$$

A few cases are described below; some other are yours to work out.

The case of charges was discussed in class. The space $\mathcal{L}_{[2]} \otimes \mathcal{L}_{[2]}$ has four basis vectors corresponding to different flavors an colors. One can guess that the vector $|2_+\rangle \otimes |2_-\rangle$ is mapped to the following superposition of states $|z\rangle$:

$$|2_+\rangle \otimes |2_-\rangle \mapsto \frac{1}{\sqrt{3}}\big(|e\rangle + \bar{\omega}|\mu\rangle + \omega|\mu^{-1}\rangle\big)$$

(where $\mu = (123)$). To check that our guess is correct, we apply the flavor and color operators corresponding to μ on both sides. On the left-hand side, we use the definition of $|2_+\rangle$, $|2_+\rangle$ (see Eq. (11)); on the right-hand side we use Eq. (5). Now we can, for example, apply A^{σ}_{col} on both sides. Thus we get:

$$|2_+\rangle \otimes |2_+\rangle \mapsto \frac{1}{\sqrt{3}}\big(|\sigma^{-1}\rangle + \bar{\omega}|\mu\sigma^{-1}\rangle + \omega|\mu^{-1}\sigma^{-1}\rangle\big).$$

One can actually do without guessing, but that requires more knowledge of representation theory. You may try this route if you are confident enough. The idea is to construct a central element of the group algebra: $\Pi_{\bar{\alpha}} = |G|^{-1} \sum_{g\in G} \chi_\alpha(g)\, g$. It belongs to the intersection of the flavor and color worlds: its actions on the flavor (obtained by replacing g with A_{fl}^g from Eq. (5)) and on the color are identical. The operator $\Pi_{\bar{\alpha}}$ is the projector onto the term $\mathcal{L}_{\bar{\alpha}} \otimes \mathcal{L}_\alpha$ in the decomposition (6).

A pure vortex $|w\rangle = |G|^{-1/2} \sum_{z\in G} |z, w\rangle$ has only color but no flavor. (It may also be regarded as a certain superposition of flavors.) Pure vortices, say, in the conjugacy class C_2 do not span the whole multiplet $\mathcal{L}_{C_2} \otimes \mathcal{L}_{C_2}$. Indeed, there are 9 states in this multiplet but only 3 pure vortices in the same class. The remaining 6 states are to be found among dyons. Basis states in this multiplet can be characterized by a pair of elements $v, w \in C_2$. For example, the particle with flavor and color $v = w = (23)$ is written in the $|z, w\rangle$ basis as follows:

$$|(23)\rangle \otimes |(23)\rangle \mapsto \frac{1}{\sqrt{2}}\big(|e,(23)\rangle + |(23),(23)\rangle\big).$$

Here, we symmertize over all z such that $zwz^{-1} = v$. But in the case of a $[C_2, -]$ dyon, we should antisymmetrize. Thus we get

$$|(23),-\rangle \otimes |(23),-\rangle \mapsto \frac{1}{\sqrt{2}}\big(|e,(23)\rangle - |(23),(23)\rangle\big).$$

Acting by, say, $A_{\text{fl}}^{(12)}$, we find another flavor-color state:

$$|(31),-\rangle \otimes |(23),-\rangle \mapsto \frac{1}{\sqrt{2}}\big(|(31),(23)\rangle - |(132),(23)\rangle\big).$$

Questions:

a) Consider an excitation with flux in the conjugacy class C_3. Its Hilbert space $\mathcal{M}$ has dimension 12, with basis vectors of the form $|z, w\rangle$, where $w \in C_3$. It can be represented as a sum of three multiplets of dimension $2 \times 2 = 4$ each:

$$\mathcal{M} = \Big(\mathcal{L}_{[C_3]} \otimes \mathcal{L}_{[C_3]}\Big) \oplus \Big(\mathcal{L}_{[C_3,\bar{\omega}]} \otimes \mathcal{L}_{[C_3,\omega]}\Big) \oplus \Big(\mathcal{L}_{[C_3,\omega]} \otimes \mathcal{L}_{[C_3,\bar{\omega}]}\Big).$$

Write out all eight flavor-color states in the first two multiplets. Identify pure vortices within $\mathcal{L}_{[C_3]} \otimes \mathcal{L}_{[C_3]}$, using the flavor-color basis.

b) Consider a pair of charges that form a color singlet. Let the flavor of both charges be $|2_+\rangle$. Write the overall state in the flavor-color basis, $|\text{flavor}_1, \text{color}_1; \text{flavor}_2, \text{color}_2\rangle$, and also in the $|z_1, w_1; z_2, w_2\rangle$ basis. Check that the neutrality conditions (8) are fulfilled (a few words of explanation will be enough).

When computing with S_3 anyons, we only use color. This is straightforward for pure vortices, since they don't have flavor. Charges and dyons come with a flavor variable attached, but we simply ignore it (because it is not protected from errors). In the following problems, we only deal with color states.

3.3 Measurement by braiding and fusion.

a) As discussed in class, the total flux of several vortices can be measured by creating a pair of non-Abelian charges from the vacuum, taking the first charge around the set of vortices, and fusing it with the second. Since the result of this procedure is probabilistic, it has to be repeated several times. The remnant particles (left of the fusion) are fused together, one by one. A good time to stop is when nothing is left (which will happen sooner or later).

   Suppose that the vortices carry flux in the conjugacy class C_3. Calculate the probability of each possible fusion outcome ([+], [−], and [2]). **Hint:** Consider a vortex of color $w = (123)$ and the charge singlet $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|2_+, 2_-\rangle + |2_-, 2_+\rangle\big)$. Taking a charge around the vortex corresponds to the action of the group element w according to Eq. (11).

b) Let us devise a procedure to distinguish a trivial charge from charge [2] without even touching it. We create a C_2 vortex singlet,

$$|\eta\rangle = \frac{1}{\sqrt{3}}\big(|(12),(12)\rangle + |(23),(23)\rangle + |(31),(31)\rangle\big),$$

   take the second vortex around the charge, and fuse the vortices. Show that in the case of trivial charge (or charge [−]) the vortices always annihilate, whereas in the case of charge [2] they form some remnant particle. What is this particle? **Hint:** Let the charge be in state $|2_+\rangle$. Apply the braiding and identify the final state of the vortex pair with some vector in a suitable irrep.

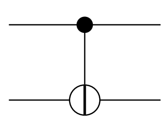
c) *(Extra credit worth* 1/2 *of a problem)* What will happen if the above-mentioned procedure is repeated twice (with the same charge but different vortex pairs), and the remnant particles fuse?

3.4 Computation using S_3 anyons. It was shown in class that one can encode a qutrit into a state of two C_2 vortices with trivial total flux:

$$\begin{aligned}
|0\rangle &\mapsto |(12),(12)\rangle,\\
|1\rangle &\mapsto |(23),(23)\rangle,\\
|2\rangle &\mapsto |(31),(31)\rangle.
\end{aligned} \tag{16}$$

Using this encoding, the following gates can be implemented.

1. The pull-trough gate: $U|w_1, w_1^{-1}, w_2, w_2^{-1}\rangle = |w_1, w_1^{-1}, w_1w_2w_1^{-1}, w_1w_2^{-1}w_1^{-1}\rangle$ or, in the qutrit basis,



$$U|a,b\rangle = |a, -a-b \bmod 3\rangle. \tag{17}$$

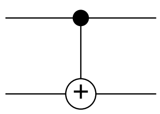
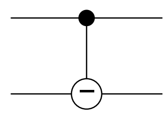
2. The creation of a vortex pair in the charge-singlet state. It corresponds to the qutrit state $|\tilde{0}\rangle$, where $|\tilde{a}\rangle$ is an element of a dual basis:

$$\begin{aligned}
|\tilde{0}\rangle &= \tfrac{1}{\sqrt{3}}\big(|0\rangle + |1\rangle + |2\rangle\big),\\
|\tilde{1}\rangle &= \tfrac{1}{\sqrt{3}}\big(|0\rangle + \omega|1\rangle + \bar{\omega}|2\rangle\big),\\
|\tilde{2}\rangle &= \tfrac{1}{\sqrt{3}}\big(|0\rangle + \bar{\omega}|1\rangle + \omega|2\rangle\big).
\end{aligned} \tag{18}$$

3. Comparison of two qutrits in the standard basis, i.e., the projective measurement corresponding to the subspace spanned by basis vectors $|a,a\rangle$ and the orthogonal complement of that subspace. (The latter is spanned by $|a,b\rangle$ such that $a \neq b$.) To implement this operation, we place two vortex pairs next to each other, $|w_1, w_1^{-1}, w_2, w_2^{-1}\rangle$ and measure the total flux of the two vortices in the middle (the flux is given by the conjugacy class of $w_1^{-1}w_2$) as described in Problem 3.3a.

4. Measurement of a qutrit with respect to $|\tilde{0}\rangle$, preserving the coherence on the orthogonal complement. To this end, we needs to tell whether the total charge of the vortex pair (i.e., some superposition of $|w, w^{-1}\rangle$) is $[+]$ or $[2]$ (see Problem 3.3b).

It was also shown that, using the comparison gate, one can establish a qutrit "bureau of standards", i.e., reference copies of $|0\rangle$ and $|1\rangle$: we name so two arbitrary distinct basis state. Then one can realize these gates:


$U_+|a,b\rangle = |a, b+a \bmod 3\rangle,$

$U_-|a,b\rangle = |a, b-a \bmod 3\rangle.$ (19)

Since $U_-\big(|\tilde{a}\rangle \otimes |\tilde{b}\rangle\big) = |\widetilde{a+b}\rangle \otimes |\tilde{b}\rangle$ (where addition *modulo* 3 is assumed), one can also copy states in the dual basis. (Take $a = 0$.) Therefore one can establish yet another standard: a reference copy of $|\tilde{1}\rangle$. The choice between $|\tilde{1}\rangle$ and $|\tilde{2}\rangle$ is again arbitrary, but it is done once and for all.

The universality of the qutrit gate set can be demonstrated by implementing the following set of qubit gates:

1. Create the state $|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$.

2. Measure in the standard basis: $|0\rangle$ vs. $|1\rangle$.

3. Measure in the dual basis: $|+\rangle$ vs. $|-\rangle$.

4. The gate σ^z.

5. The gates $\Lambda^2(-1)$ and $\Lambda^3(-1)$ (the latter takes $|x,y,z\rangle$ to $(-1)^{xyz}|x,y,z\rangle$).

Note that the measurement in the standard basis follows immediately from the ability to compare qutrits with 0 and 1. The creation of $|+\rangle$ is also simple: we perform the comparison with 2 on the quantum state $|\tilde{0}\rangle$. If the answer is "no", we have obtained $|+\rangle$ as the result of projection, otherwise we should try again with a new copy of $|\tilde{0}\rangle$. Let us try to implement the other qubit gates.

Questions:

a) Consider a sequence of measurements: we compare a qutrit with $|\tilde{0}\rangle$ and with 2 alternately, until we get a "yes" (or get tired). How does this procedure help us to distinguish $|+\rangle$ from $|-\rangle$? Find the error probability if we stop after n negative answers.

b) Using the qutrit gates, construct the ancillary state

$$|\xi\rangle = \frac{1}{\sqrt{3}}\big(|0\rangle - |1\rangle + |2\rangle\big). \tag{20}$$

Hint: First, figure how to implement this gate: $T|a\rangle = \omega^a|a\rangle$. Then, take two copies of $|+\rangle$ and turn them into

$$|\eta\rangle = \frac{1}{2}\Big(|0\rangle + \omega|1\rangle\Big) \otimes \Big(|0\rangle + \bar{\omega}|1\rangle\Big).$$

Apply the gate U_+ and compare the first qutrit with $|\tilde{0}\rangle$. Compute the state of the second qutrit, assuming that the answer was "yes". To this end, you need to calculate $\big(\langle\tilde{0}| \otimes I\big)U_+|\eta\rangle$.

c) *(Extra credit worth* 1/2 *of a problem)* Show how to flip the sign of c_0, c_1, or c_2 in the superposition $|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$. (The flipping of c_1 is reduced to σ^z if we work with qubits.) **Hint:** Apply the gate U_+ to $|\psi\rangle \otimes |\xi\rangle$, where $|\xi\rangle$ is the ancilla defined in Eq. (20), and measure the second qutrit in the standard basis.

d) *(Extra credit worth* 1/2 *of a problem)* Implement the gates $\Lambda^2(-1)$ and $\Lambda^3(-1)(\sigma^z)^{\otimes 3}$. **Hint:** Use addition/subtraction *modulo* 3 and the sign flip.