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LIMITATIONS ON THE STATISTICAL DESCRIPTION OF BLACK HOLES

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We argue that the description of a black hole as a statistical (thermal) object must break down as the extreme (zero-temperature) limit is approached, due to uncontrollable thermodynamic fluctuations. For the recently discovered charged dilaton black holes, the analysis is significantly different, but again indicates that a statistical description of the extreme hole is inappropriate. These holes invite a more normal elementary particle interpretation than is possible for Reissner–Nordström holes.

1. Introduction

The extreme Kerr–Newman black holes have zero temperature but non-zero entropy.[1] This makes them a very intriguing subject for investigation. Non-zero entropy at zero temperature normally indicates a degenerate ground state. Thus arises the challenge of understanding the nature of these degenerate states.

In contrast, the classical no-hair theorems[2] indicate that there is no degeneracy at the classical level; for each value of the macroscopic parameters M, Q, and J, there is a unique classical black-hole configuration. Presumably, one could probe the possible states of an extreme black hole by constructing the quantum-mechanical S-matrix for scattering of elementary quanta from the hole. The non-vanishing entropy suggests that a large matrix is required, for the scattering process may change the "internal" state of the black hole.

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There is another possible description of scattering from extreme black holes, using very different concepts. One might argue that when a particle impinges on the black hole it raises the temperature of the hole. The hole subsequently radiates as a black (or gray) body of that temperature. This thermodynamic description of the scattering process does not discriminate among microscopic internal states of the hole.

We argue here that the thermal description of a black hole becomes ill-defined (and must be replaced by a significantly different quantum-mechanical description) as the black hole approaches the extreme limit. We also analyze the recently discovered[3,4] charged dilaton black holes. The thermodynamic behavior of the dilaton black hole is quite different from that of the Kerr–Newman black hole—the extreme black holes have zero entropy and non-zero temperature, rather than the other way around. But our analysis reveals limitations on the thermal description in this case as well.

2. Extreme Kerr–Newman Black Holes

The standard semiclassical treatment of black hole radiance[5] neglects the back reaction of the emitted radiation on the hole. This approximation is not self-consistent if the emission of a typical quantum of radiation changes the temperature by an amount comparable to the value of the temperature. In that event, one does not know whether to use the temperature before emission, the temperature after emission, or something in between when calculating the Boltzmann factor governing the emission.

To determine under what conditions the usual thermal treatment is appropriate, we recall the basic equations of black hole thermodynamics.[1] The temperature of a black hole with mass M, charge Q, and angular momentum J is $T = \kappa/2\pi$, where κ is the surface gravity; it may be expressed as

$$8\pi MT = \frac{2f^{1/2}}{1 - \frac{1}{2}(Q^2/M^2) + f^{1/2}}\,, \tag{1}$$

where

$$f \equiv 1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}\,. \tag{2}$$

(Our units are chosen so that $\hbar = c = G = 1$.) The entropy is $S = \frac{1}{4}A$, where A is the area of the event horizon, or

$$S = 2\pi M^2\left(1 - \frac{1}{2}(Q^2/M^2) + f^{1/2}\right). \tag{3}$$

The Kerr–Newman solution actually has an event horizon (and so is a black hole rather than a naked singularity) only for $f \geq 0$. The extreme black holes, for which the temperature vanishes, satisfy $f = 0$. They have finite entropy $S \geq \pi M^2$.

Under the assumption that the typical emitted quantum carries energy T but no charge or angular momentum (an assumption that we reconsider below), the condition for the thermal description to be self-consistent is

$$\left| T \left(\frac{\partial T}{\partial M} \right)_{Q,J} \right| \ll |T| \,. \tag{4}$$

From Eq. (1), we find

$$\left(\frac{\partial T}{\partial M} \right)_{Q,J} \simeq \frac{1}{2\pi M^2} f^{-\frac{1}{2}} \tag{5}$$

when $f \simeq 0$. Thus the thermal description of the black hole breaks down when $M^2 f^{\frac{1}{2}} \lesssim 1$.

Note that this breakdown may occur within the regime $M^2 \gg 1$. In this regime, the curvature at the horizon is small in Planck units, and corrections due to quantum gravity are expected to be negligible. On the other hand, if we had applied our argument to a Schwarzschild black hole, $Q = J = 0$, we would have found that the thermal description breaks down at the same time as the classical description of space-time, namely for $M^2 \sim 1$.

It is instructive to consider the condition Eq. (4) from the point of view of thermodynamics. Using the first law of thermodynamics, we may restate Eq. (4) as

$$T \left(\frac{\partial S}{\partial T} \right)_{Q,J} \gg 1 \,. \tag{6}$$

Thus our requirement is that the *available entropy* of the hole should be much larger than unity. The statistical treatment of the radiation is inappropriate if the ensemble of states from which it is drawn is small. In estimating the size of this ensemble, we should not include the residual entropy at zero temperature, since this is unavailable to the radiation.

Equation (5) also says that the heat capacity $C_{Q,J} \equiv (\partial M/\partial T)_{Q,J}$ tends to zero as the extreme limit is approached. Standard thermodynamic arguments[6] relate the heat capacity to fluctuations, both in temperature and in entropy. For the fluctuations in temperature one has

$$\langle (\Delta T)^2 \rangle / T^2 = 1/C \,. \tag{7}$$

Thus as $C \to 0$, the temperature fluctuations become large compared to the temperature itself. For the entropy fluctuations one has

$$\langle (\Delta S)^2 \rangle = C \,. \tag{8}$$

Thus when $C \sim 1$ few states are sampled spontaneously, as anticipated above.

Of course, the third law of thermodynamics asserts that the heat capacity of *any* finite system vanishes at zero temperature, and so these remarks about fluctuations in temperature and entropy apply very generally. Thermodynamics describes the behavior of a system at arbitrarily low temperature only if the system is arbitrarily large. It is significant that a black hole is no exception to this rule, for it follows that back reaction effects become significant for a sufficiently cold black hole of given mass.

To complete this discussion, let us re-examine the assumption underlying the condition Eq. (4), namely that the typical emitted quantum carries energy of order T and no charge or angular momentum. As T approaches zero, the wavelength of a quantum with energy T becomes very large compared to the size of the black hole. The resulting impedance mismatch favors the emission of more energetic quanta. But this effect only strengthens our argument, for the condition Eq. (4) becomes replaced by a (slightly) more restrictive condition.

Now consider the charge and angular momentum carried away by the emitted radiation. A charged black hole tends to discharge as it radiates, and a spinning black hole tends to spin down. Fluctuations are therefore controlled, not by the heat capacity $C_{Q,J}$, but by an effective heat capacity that takes these tendencies into account. If the radiation emitted changes Q and J, then Eq. (5) is replaced by

$$\delta T \simeq \frac{1}{2\pi M^2} f^{-\frac{1}{2}} \left[\delta M - \frac{1}{1+J^2/M^4}\left(\frac{Q}{M}\delta Q + \frac{J}{M^2}\frac{\delta J}{M}\right)\right], \tag{9}$$

where $f \simeq 0$. The effective heat capacity $\delta M/\delta T$ becomes small provided that $\delta Q/\delta M$ and $\delta J/M\delta M$ obey suitable upper bounds, which ought to be imposed on other grounds. The condition on $\delta Q/\delta M$ prevents the extreme hole from discharging due to dielectric breakdown of the vacuum outside the event horizon; it is satisfied if the charge-to-mass ratios of all elementary particles are sufficiently small.[7] The condition on $\delta J/M\delta M$ prevents the extreme hole from spinning down due to spontaneous emission of superradiant modes; it is satisfied if the hole is enclosed in a sufficiently small cavity.[8]

3. Charged Dilaton Black Holes

A new family of exact black hole solutions has been discovered recently.[3,4] These solve the Einstein equations for a model of gravity coupled to a Maxwell field and to a massless scalar "dilaton" field, a model that arises as a low-energy approximation to superstring theory.[9] The action for this model is (in the notation of Ref. 4)

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,(-R + 2(\nabla\phi)^2 + e^{-2\phi}F^2). \tag{10}$$

Both magnetically charged and electrically charged solutions were found. The magnetically charged solution may be written in the form

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + R^2 d\Omega\,, \tag{11}$$

$$e^{-2\phi} = \frac{R^2}{r^2} = 1 - \frac{Q^2}{Mr}, \quad F = Q \sin\theta \, d\theta \wedge d\phi . \tag{12}$$

The electrically charged solution has the same geometry, but the dilaton and Maxwell fields are given by

$$e^{-2\phi} = \frac{r^2}{R^2}, \quad e^{-2\phi} F = \frac{Q}{R^2} \, dt \wedge dr . \tag{13}$$

These solutions have an event horizon at $r = 2M$, and are regular outside the horizon for $Q^2 < 2M^2$. When this inequality is violated, there is a naked singularity at $r = Q^2/M$.

The dilaton field ϕ is excited outside the horizon, but the "dilaton charge" is not an independent quantum number of the hole. The behavior of the dilaton is completely dictated by M, Q, and the "vacuum" value ϕ_0 attained by ϕ at $r = \infty$. We have set $\phi_0 = 0$ by absorbing a factor $e^{-\phi_0}$ into the normalization of the Maxwell field.

The temperature of the hole can be inferred from the periodicity of its Euclidean continuation,[10,11] or alternatively from its surface gravity. It is $T = 1/8\pi M$.

The entropy can be evaluated in three ways (one of which leaves it undetermined by a multiplicative constant). Following Bekenstein,[12] one can argue *a priori* that the entropy must be proportional to the surface area of the hole. Alternatively, knowing the temperature and chemical potential, one may integrate the first law of thermodynamics to find the entropy. (The chemical potential is the derivative of the Coulomb energy with respect to Q.) Finally one may calculate the thermodynamic functions directly, by exhibiting the black hole (continued to imaginary time) as a saddle point contribution to the partition function.[10] In the present case, all three methods of evaluation agree, to give

$$S = 4\pi M^2 \left(1 - \frac{1}{2}(Q^2/M^2)\right) . \tag{14}$$

(The area of the event horizon, unlike the Euclidean action, is not invariant under dilaton-dependent conformal rescalings of the metric. Indeed, it has been argued that $e^{2\phi} g_{\mu\nu}$ is the natural metric in string theory,[9,13] and with this choice we would obtain a different expression for the Bekenstein entropy. We have used the "canonical" metric for which the action is Eq. (10), for several reasons.[14] With this choice, mixing of ϕ and g is removed, inertial and gravitational mass coincide, and the weak energy condition is satisfied, so that Hawking's area theorem[15] applies.)

A striking feature of the thermodynamics is that the entropy goes to zero at a finite temperature for the extreme hole. This is quite different from (in fact, the reverse of) the result for the extreme Reissner–Nordström hole. In that case, the temperature goes to zero but the entropy remains finite. (In both cases, the luminosity due to Hawking radiation approaches zero.)

The curvature at the event horizon of the near-extreme dilaton black hole is large; this raises the issue of quantum gravity corrections, to which we return below. But

Eq. (14), taken seriously, invites a more normal "elementary particle" interpretation than is possible for extreme Reissner–Nordström holes. Indeed, if the entropy is interpreted as a measure of the number of available states, the fact that it goes to zero at a non-zero temperature means that there is an effective *mass gap* of order $T = 1/8\pi M$ splitting the extreme black hole ground state from its lowest excitation.

This idea may seem paradoxical, since one might imagine that by throwing very soft quanta (e.g. soft photons) into the hole one could create an abundance of accessible low-energy states. However, two facts about the hole make it plausible that these putative states do not really exist. First, since the area of the event horizon goes to zero, the classical capture cross-section vanishes. Thus the attempt to "throw in" soft quanta will generally not lead to additional black hole states, but to scattering states of quantum plus black hole. Second, when quantum corrections are included, absorption of sufficiently energetic incoming quanta will surely occur. But even if we succeed in injecting a quantum of energy $\delta M \sim T$ into the hole, the resulting object will have nominal entropy only $\delta S \sim \delta M/T \sim 1$. Since the thermal description of black hole emission is applicable only for $S \gg 1$, we cannot conclude that the putative black hole state is distinguishable from a scattering state, nor that it is long-lived.

One anticipates on general grounds, however, that the semiclassical approximation used to compute Eq. (14) must break down by the time $S \sim 1$. Indeed, the heat capacity of a dilaton black hole is $C_Q = -1/8\pi T^2$, as for a Schwarzschild black hole. Because the heat capacity is negative, equilibrium thermodynamics properly applies only to a hole in equilibrium with radiation in a sufficiently small cavity. In such a system, the black hole entropy inevitably undergoes fluctuations with $\Delta S \sim (-C_Q)^{1/2} \sim 1/T$. Thus we expect that the corrections to the semiclassical calculation of the entropy will be significant for $S \sim M \gg 1$.

To estimate the corrections, we must appeal to a model of quantum gravity. It is especially instructive to contemplate corrections to Eq. (10) that arise in string theory. Two types of corrections need to be considered—corrections in classical string theory that are higher order in α', and quantum corrections that are higher order in string loops. Interestingly, for a near-extreme dilaton black hole, quantum corrections to the entropy are dominant in the magnetically charged case, while classical corrections are more important in the electrically charged case.

Let us consider the electrically charged case more closely. After we do the conformal rescaling to put the metric in canonical form, the α' expansion becomes an expansion in $\alpha' e^{-2\phi}$; in the corrections to the action, powers of this parameter accompany powers of F and of the curvature. A dimensionless quantity that controls the expansion can be identified by plugging Eqs. (11) and (13) for a near-extreme hole into the corrected action. (Powers of the curvature behave like powers of R^{-2}.) One finds that the expansion parameter is

$$\varepsilon_{\text{electric}} = \frac{\alpha'(GM)^2}{R^4}\,. \tag{15}$$

In the electrically charged case, then, the expansion breaks down for $R^2 \lesssim GM(\alpha')^{1/2}$. In terms of thermodynamic quantities, we find that the expansion in α' breaks down for $S \sim (\alpha')^{1/2}/T$—just as anticipated above. As these large corrections may be viewed as an unavoidable consequence of thermodynamic fluctuations, they do not necessarily invalidate the suggestion that a finite gap separates the black hole ground state from its lowest excitation.

(It is possible, though, that this perturbative analysis is seriously misleading for the full theory, especially for large black holes. Indeed, if the theory is to be realistic we must anticipate that the dilaton acquires a mass through non-perturbative effects. In that case, the picture sketched above could apply only for black holes that have large curvature on the scale of the dilaton Compton wavelength. Furthermore, a realistic theory contains light charged particles, so that extreme holes discharge rapidly due to dielectric breakdown.)

In the magnetic case the effective expansion parameter for the classical theory is

$$\varepsilon_{\text{magnetic}} = \frac{\alpha'}{(GM)^2}, \tag{16}$$

which is well behaved for large M. However, the string loop expansion, governed by $e^{2\phi}$, breaks down near the horizon.

4. Modified Dilaton Couplings

The authors of Refs. 3 and 4 also considered a modified form of the dilaton coupling, where $e^{-2\phi}$ is replaced by $e^{-2a\phi}$ in Eq. (10), and found a family of solutions parametrized by a. It is instructive to compare the thermodynamic properties of the various solutions in this family, especially the near-extreme ones. (This was also discussed in Ref. 3.)

The temperature and entropy are

$$T = \frac{1}{4\pi r_+}\left(\frac{r_+ - r_-}{r_+}\right)^{\frac{1-a^2}{1+a^2}}, \quad S = \pi r_+^2 \left(\frac{r_+ - r_-}{r_+}\right)^{\frac{2a^2}{1+a^2}}. \tag{17}$$

Here $r_\pm$ are determined in terms of M and Q according to $M = \frac{r_+}{2} + \left(\frac{1-a^2}{1+a^2}\right)\frac{r_-}{2}$, $Q^2 = \left(\frac{r_+ r_-}{1+a^2}\right)$. For the extreme holes $r_+ - r_- \to 0^+$.

It is interesting that the entropy approaches 0 in the extreme limit for any case *except* $a = 0$, which is the standard Reissner–Nordström case. This is in keeping with the notion that the large degeneracy of the extreme Reissner–Nordström black hole is quite special, and is unstable with respect to generic perturbations.

As for the temperature, the case $a = 1$ appears to be special. Only in that case does the extreme hole have a non-zero finite temperature. For $a > 1$, the temperature actually blows up. But for any finite a, TR, where $R \equiv \sqrt{S/\pi}$ is the physical radius, approaches zero as $r_+ - r_-$ approaches zero; thus the thermal wavelength gets very small compared to the size of the hole in the extreme limit.

Since there is a severe impedance mismatch for the potential radiation, it is quite plausible that evaporation will stop, even if the (formal) temperature blows up.

Of course, one anticipates that semiclassical theory breaks down for a sufficiently small black hole. If a is not too close to one, then the heat capacity is $|C_Q| \sim S$, and fluctuations in temperature and entropy may be reasonably small as long as $S \gg 1$ (which is required in any case for a classical picture of space-time to apply). Taking $S \sim 1$ in Eq. (17), we find $T \sim M^{-1/a^2}$ for a Planck-size black hole. Thus, the distinction between $a < 1$ and $a > 1$ may not be so dramatic as Eq. (17) seems to suggest.

5. Conclusion

We have argued that the description of black holes as thermal objects must break down as the extreme limit is approached. For Kerr–Newman black holes, temperature fluctuations become large in this limit; for charged dilaton black holes, entropy fluctuations become large. In both cases, scattering of quanta by the black hole *cannot* be described as classical absorption followed by thermal emission; rather, the scattering process can be accurately described only if gravitational back reaction effects are consistently included. The extreme charged dilaton black holes plausibly may be regarded as isolated non-degenerate states in the spectrum of elementary particles.

Our main conclusions might be constructed negatively: extreme black holes can not be well modeled as macroscopic thermal bodies, nor are their qualitative properties independent of the nature of the non-gravitational interactions. However, they might also be interpreted positively, as an invitation to construct and analyze models where the behavior of the extreme holes is by some criterion "reasonable"—e.g., manifestly consistent with the standard rules of quantum mechanics, or resembling the properties of elementary particles.

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