

Complementarity in wormhole chromodynamics ☆

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The electric charge of a wormhole mouth and the magnetic flux “linked” by the wormhole are non-commuting observables, and so cannot be simultaneously diagonalized. We use this observation to resolve some puzzles in wormhole electrodynamics and chromodynamics. Specifically, we analyze the color electric field that results when a colored object traverses a wormhole, and we discuss the measurement of the wormhole charge and flux using Aharonov–Bohm interference effects.

1. Introduction

Many years ago, Wheeler [1] and Misner and Wheeler [2] proposed that electric field lines trapped in the topology of a multiply-connected space might explain the origin of electric charge. Consider a three-dimensional space with a handle (or “wormhole”) attached to it, where the cross section of the wormhole is a two-sphere. On this space, the source-free Maxwell equations have a solution with electric field lines caught inside the wormhole throat. One mouth of the wormhole, viewed in isolation by an observer who is unable to resolve the small size of the mouth, cannot be distinguished from a pointlike electric charge. Only when the observer inspects the electric field more closely, with higher resolution, does she discover that the electric field is actually source-free everywhere.

It is also interesting to consider what happens when a charged particle traverses a wormhole^{#1}. (Of course, this “pointlike” charge might actually be one mouth of a smaller wormhole.) Suppose that, initially, the mouths of the wormhole are uncharged (no electric

flux is trapped in the wormhole). By following the electric field lines, we see that after an object with electric charge Q traverses the wormhole, the mouth where it entered the wormhole carries charge Q , and the mouth where it exited carries charge $-Q$. Thus, an electric charge that passes through a wormhole transfers charge to the wormhole mouths.

In this note, we will address two (closely related) puzzles associated with this type of charge transfer process. Our first puzzle concerns the quantum mechanics of charged particles in the vicinity of a wormhole. We can compute the amplitude for the particle to propagate from an initial position to a final position by performing a sum over histories. Naively, one would expect this sum to include histories that traverse the wormhole, and that the contribution to the path integral due to these histories should be combined coherently with the contribution due to histories that do not traverse the wormhole. In fact, the histories can be classified according to their “wind-

^{#1} Note that we are assuming that the wormhole is traversable. This assumption would be valid for a non-dynamical wormhole three-geometry, but it is in conflict with the “topological censorship” theorem [3] that can be proved in classical general relativity (with suitable assumptions about the positivity of the energy–momentum tensor). The traversability of the wormhole might be enforced by quantum effects. Alternatively, the reader might prefer to envision our space as a thin two-dimensional film, containing objects with Aharonov–Bohm interactions. Such wormholes might actually be fashioned in the laboratory!

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ing number" around the wormhole, which can take any integer value, and one expects that all of the winding sectors should be combined coherently. Upon further reflection, though, one sees that, for charged particles, this naive expectation must be incorrect. Long after the final position of the particle has been detected, an observer can measure the charge of one of the wormhole mouths. If the mouth was uncharged initially, and carries charge nQ finally, then the observer concludes that the charged particle must have entered that mouth of the wormhole n times. Because the winding sectors are perfectly correlated with the charge transferred to the mouth, the amplitudes associated with different numbers of windings cannot interfere with one another. The puzzle in this case is to understand more clearly the mechanism that destroys the coherence of the different winding sectors.

Our second puzzle arises in a non-abelian gauge theory, such as quantum chromodynamics. Suppose that a wormhole initially carries no color charge, and consider what happens when a "red" quark traverses the wormhole. (We can give a gauge-invariant meaning to the notion that the quark is red by establishing a "quark bureau of standards" at some preferred location, and carefully preserving a standard red (R) quark, blue (B) quark, and yellow (Y) quark there. When we say that a quark at another location is red, we mean that if it is parallel transported back to the bureau of standards, its color matches that of the standard R quark. This notion is especially simple if we assume that there are no color magnetic fields, so that parallel transport is unaffected by smooth deformations of the path.) An observer who watches the red quark enter one mouth of the wormhole concludes that the mouth becomes a red source of color electric field^{#2}. But the other mouth of the wormhole is initially in a color-singlet state, and it cannot suddenly acquire a long-range color electric field as the quark emerges from the mouth. Thus, after the traversal, the quark and mouth must be in the color-singlet state

$$\frac{1}{\sqrt{3}} (|R\rangle_{\text{quark}} \otimes |\bar{R}\rangle_{\text{mouth}} + |B\rangle_{\text{quark}} \otimes |\bar{B}\rangle_{\text{mouth}} + |Y\rangle_{\text{quark}} \otimes |\bar{Y}\rangle_{\text{mouth}}) . \quad (1)$$

The puzzle in this case is to understand why the quark that emerges from the wormhole is not simply in the color state $|R\rangle$, and how the correlation between the color of the quark and the color of the mouth is established.

The resolution of these puzzles involves some peculiar features of the Aharonov-Bohm effect [4] on non-simply connected manifolds. The essential concept is the magnetic flux "linked" by the wormhole. If a particle with charge Q is carried around a closed path that traverses a wormhole (in a $U(1)$ gauge theory), it in general acquires an Aharonov-Bohm phase $\exp(iQ\Phi)$, where Φ is the flux associated with the path. (This flux is defined modulo the flux quantum $\Phi_0 = 2\pi/e$, where e is the charge quantum.) If magnetic field strengths vanish everywhere, this flux is a topological invariant, unchanged by smooth deformations of the path. The crucial point is that the flux Φ and the charge of a wormhole mouth are complementary observables – if the mouth has a definite charge (like zero), then the flux does not take a definite value. Summing over the different possible values of the flux generates the decoherence of the winding sectors described above, and also (in the non-abelian case) causes the red quark that traverses the wormhole to emerge in the state eq. (1).

2. Wormhole complementarity

Let us now analyze these Aharonov-Bohm interactions in greater detail. We will use a notation that is appropriate when the gauge group G is a finite group. This will serve to remind the reader that our analysis applies to the case of a *local discrete symmetry* [5–7]. For the case of a continuous gauge group, one need only replace sums by integrals in some of the expressions below. When the gauge group is discrete (and also when it is continuous), the electric charge of an object, including a wormhole mouth, can be measured in principle by scattering a loop of *cosmic string* (or a closed magnetic solenoid) off of the object. For ease of visualization, we will carry out our explicit analysis for the case of two spatial di-

^{#2} We are assuming that the wormhole is being examined on a sufficiently short distance scale that the effects of color confinement can be neglected.

mensions, so that charges are measured by scattering magnetic vortices. The analysis in three spatial dimensions is similar.

There are actually two types of topological magnetic flux associated with a wormhole in two spatial dimensions, for there are two topologically distinct paths for which Aharonov–Bohm phases can be measured, as shown in fig. 1. The path α encloses one mouth of the wormhole, and we will denote the group element associated with parallel transport around this path as $a \in G$. The path β passes through both wormhole mouths, and we denote the associated group element as $b \in G$. We refer to these group elements as the α -flux and β -flux of the wormhole, and denote the corresponding quantum state of the wormhole as $|a, b\rangle_{\text{wormhole}}$.

(In three spatial dimensions, we may wish to consider a wormhole that has the topology of $M \times R$, where M is a Riemann surface. In the case most commonly of interest, M is a two-sphere; in that case, there is no non-contractible path that “wraps around” one mouth of the wormhole, and hence there is no topological α -flux.)

Now, we can measure the electric charge of a wormhole mouth by winding a vortex around the mouth, and observing the Aharonov–Bohm phase acquired by the vortex. However, winding the vortex around the mouth will also change the state $|a, b\rangle$ of the wormhole. For our purposes, it will be sufficient to consider the special case in which $a = e$, the identity. (A more general analysis of non-abelian Aharonov–Bohm interactions on topologically nontrivial spaces can be found in ref. [8].) As shown in fig. 2, we may enclose the vortex with a closed path γ ; we denote the group element associated with transport around γ as $h \in G$, and refer to it as the flux of the vortex. As the



Fig. 1. Two non-contractible paths α (a) and β (b), beginning and ending at an arbitrarily chosen basepoint x_0 , on the wormhole geometry. The group elements associated with parallel transport around these paths are the α -flux and β -flux of the wormhole.

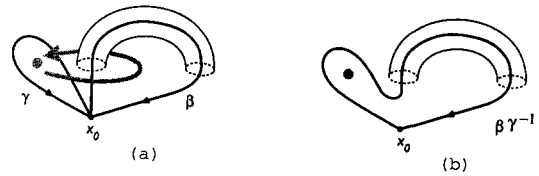


Fig. 2. A vortex winds around one mouth of the wormhole, as shown in (a). If the path $\beta\gamma^{-1}$ shown in (b) is deformed during the winding of the vortex, so that the vortex never crosses the path, $\beta\gamma^{-1}$ evolves to the path β .

vortex winds counterclockwise around the wormhole mouth, the path $\beta\gamma^{-1}$ is deformed to β . (Here, $\beta\gamma^{-1}$ denotes the path that is obtained by tracing γ^{-1} first, followed by β .) Thus, when the vortex winds around the mouth, the flux associated with $\beta\gamma^{-1}$ before the winding becomes the flux associated with β after the winding; we conclude that the state of wormhole and vortex is modified according to

$$|e, b\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \rightarrow |e, bh^{-1}\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \quad (2)$$

Eq. (2) is the centerpiece of our analysis. It says that if the wormhole is in the “flux eigenstate” $|e, b\rangle$, then any attempt to use Aharonov–Bohm interference to measure the electric charge of one mouth is doomed to failure. If we scatter a vortex off of the mouth (with vortex flux $h \neq e$), whether the vortex passed to the left or the right of the mouth is perfectly correlated with the state of the wormhole, and therefore no interference is seen; the probability distribution of the scattered vortex is the incoherent sum of the probability distributions for vortices that pass to the left and pass to the right.

However, by superposing the wormhole states of definite β -flux, we can construct states with definite charge. (We need only decompose the regular representation of G into irreducible representations.) In particular, in the state

$$|0\rangle_{\text{wormhole}} = \frac{1}{\sqrt{n_G}} \sum_{b \in G} |e, b\rangle_{\text{wormhole}} \quad (3)$$

(where n_G is the order of the group G), each mouth of the wormhole has zero charge. To see this, consider carrying the h -vortex around one mouth of this wormhole. It is easy to see that the state of the wormhole is unmodified, so that the Aharonov–Bohm

phase acquired by the vortex is trivial. On the other hand, suppose that we try to measure the β -flux of the wormhole by carrying a charged particle along the path β . Let us denote the initial state of the particle as $|v\rangle_{\text{particle}}$, and let (ν) be the irreducible representation of G according to which the state transforms. Then if we carry this particle around the path β where the wormhole is initially in the state $|0\rangle_{\text{wormhole}}$, the state of particle and wormhole is modified according to

$$\begin{aligned}
 |\text{initial}\rangle &\equiv |v\rangle_{\text{particle}} \otimes |0\rangle_{\text{wormhole}} \\
 &\rightarrow |\text{final}\rangle \\
 &\equiv \frac{1}{\sqrt{n_G}} \sum_{b \in G} D^{(\nu)}(b) |v\rangle_{\text{particle}} \otimes |e, b\rangle_{\text{wormhole}},
 \end{aligned}
 \tag{4}$$

thus the overlap of the final state with the initial state is

$$\begin{aligned}
 \langle \text{final} | \text{initial} \rangle &= \frac{1}{n_G} \sum_{b \in G} \langle v | D^{(\nu)}(b) | v \rangle \\
 &= 1, \quad \text{if } (\nu) = \text{trivial}, \\
 &= 0, \quad \text{otherwise}.
 \end{aligned}
 \tag{5}$$

Unless (ν) is trivial, the state of the particle that has been carried through the wormhole is orthogonal to the original state. Hence we recover our earlier conclusion that, for charged particles propagating on the wormhole geometry, paths that traverse the wormhole add incoherently with paths that do not.

We see that the wormhole cannot simultaneously have a definite β -flux and a definite charge. We call this property ‘‘wormhole complementarity’’. It is intimately related to the complementarity connection between magnetic and electric flux that was first emphasized by ’t Hooft [9], and was generalized to the non-abelian case in ref. [10].

By decomposing the regular representation eq. (2) into irreducible representations, we obtain states in which the wormhole *mouth* has a definite charge. The charge of a mouth should not be confused with the ‘‘Cheshire charge’’ [6,11] carried by the whole wormhole. To measure the charge of the whole wormhole, we would wind a vortex around *both* mouths of the wormhole. In this process, the state of vortex and wormhole is modified according to [8]

$$\begin{aligned}
 &|a, b\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \\
 &\rightarrow |hah^{-1}, hbh^{-1}\rangle_{\text{wormhole}} \\
 &\otimes |h(aba^{-1}b^{-1})h(aba^{-1}b^{-1})^{-1}h^{-1}\rangle_{\text{vortex}}.
 \end{aligned}
 \tag{6}$$

Note that $aba^{-1}b^{-1}$ is the ‘‘total flux’’ of the wormhole, the flux associated with a path that encloses both mouths. Charge measurement is possible only if the initial and final vortex states are not orthogonal, so that interference can occur. Therefore, the flux h of the vortex must commute with the total flux of the wormhole – the charge that can be detected is actually a representation of $N(aba^{-1}b^{-1})$, the centralizer of the total flux [12,6,11]. States of definite Cheshire charge are obtained by decomposing the wormhole states $|a, b\rangle$ into states that transform irreducibly under the action eq. (6), where $h \in N(aba^{-1}b^{-1})$.

Of course, to an observer with poor resolution, the wormhole mouths look like pointlike particles, and the Cheshire charge of the wormhole coincides with the Cheshire charge of vortex pairs that has been discussed elsewhere [10,13,14]. For example, if $b=e$ then the mouths appear to be a vortex with flux a and an anti-vortex with flux a^{-1} . In the case $a=e$ that we have considered, neither wormhole mouth carries any flux, and the states $|e, b\rangle_{\text{wormhole}}$ are transformed as

$$\begin{aligned}
 &|e, b\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \\
 &\rightarrow |e, hbh^{-1}\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}}
 \end{aligned}
 \tag{7}$$

when the vortex winds around the wormhole. The states of definite Cheshire charge are obtained by superposing the flux eigenstates $|e, b\rangle_{\text{wormhole}}$, with b taking values in a particular conjugacy class of G . Specifically the states

$$|0, [b]\rangle_{\text{wormhole}} = \frac{1}{\sqrt{n_{[b]}}} \sum_{b' \in [b]} |e, b'\rangle_{\text{wormhole}} \tag{8}$$

(where $[b]$ denotes the class containing b , and $n_{[b]}$ is the order of that class) have trivial total charge, although each wormhole mouth carries charge in these states.

The peculiar behavior we found for Aharonov–Bohm scattering off of a wormhole mouth, when the wormhole is in a flux eigenstate, can be given a more

conventional interpretation if we think of the wormhole as a pair of charged particles in a particular (correlated) state. For example, the flux eigenstate $|e, e\rangle_{\text{wormhole}}$ can be decomposed as

$$\begin{aligned}
 |e, e\rangle_{\text{wormhole}} &\equiv |0, [e]\rangle_{\text{wormhole}} \\
 &= \sum_{\nu} C_{\nu} \sum_i \frac{1}{\sqrt{n_{\nu}}} |e_i, \nu\rangle \otimes |e_i^*, \nu\rangle, \\
 \sum_{\nu} |C_{\nu}|^2 &= 1, \tag{9}
 \end{aligned}$$

where the $|e_i, \nu\rangle$ are a basis for the space on which the irreducible representation (ν) acts, and n_{ν} is the dimension of this representation. This is a superposition of states in which the two particles (the mouths) have nontrivial charges, and are in a combined state of trivial charge. Experiments involving one of the mouths are described by a mixed density matrix of the form

$$\rho = \sum_{\nu} |C_{\nu}|^2 \frac{1}{n_{\nu}} \mathbf{1}_{\nu}, \tag{10}$$

and Aharonov–Bohm scattering of the h -vortex off the mouth enables us to measure

$$\begin{aligned}
 \text{tr } D(h)\rho &= \sum_{\nu} |C_{\nu}|^2 \frac{1}{n_{\nu}} \chi^{(\nu)}(h) \\
 &= 1, \quad h=e, \\
 &= 0, \quad \text{otherwise,} \tag{11}
 \end{aligned}$$

where $\chi^{(\nu)}$ denotes the character of the representation. (The second equality in eq. (11) follows from the property eq. (2).) From the group orthogonality relations, we see that $|C_{\nu}|^2 = n_{\nu}^2/n_G$. Thus Aharonov–Bohm scattering enables us to determine the probability that the wormhole mouth carries charge (ν), but does not determine the relative phases of the C_{ν} [13]. When we think of it as a point particle, the unusual thing about a wormhole mouth is that it is natural to consider a state such that the mouth is in a superposition of particle states with different gauge charges.

3. Charge transfer

Now let us suppose that, after the wormhole in the initial state $|0\rangle_{\text{wormhole}}$ is traversed by the charged

particle in the initial state $|v\rangle_{\text{particle}}$, we attempt again to measure the charges of the two mouths. If an h -vortex is carried around the mouth that the charged particle *entered*, then the state of wormhole, particle, and vortex is modified according to

$$\begin{aligned}
 &\frac{1}{\sqrt{n_G}} \sum_{b \in G} D^{(\nu)}(b) |v\rangle_{\text{particle}} \\
 &\otimes |e, b\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \\
 &\rightarrow \frac{1}{\sqrt{n_G}} \sum_{b \in G} D^{(\nu)}(b) |v\rangle_{\text{particle}} \\
 &\otimes |e, bh^{-1}\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}}, \tag{12}
 \end{aligned}$$

so that the overlap of the initial state with the final state is

$$\begin{aligned}
 \text{overlap} &= \frac{1}{n_G} \sum_{b, b' \in G} \langle v | D^{(\nu)}(b')^{-1} D^{(\nu)}(b) | v \rangle \\
 &\cdot \langle e, b' | e, bh^{-1} \rangle \\
 &= \langle v | D^{(\nu)}(h) | v \rangle. \tag{13}
 \end{aligned}$$

This is exactly the same as the overlap we would have obtained if the vortex had been carried around the initial charged particle. Thus, as we anticipated, the charge of the particle has been transferred to the mouth of the wormhole.

But if we measure instead the charge of the *other* mouth, we obtain a rather different result. It is actually most instructive to consider carrying the h -vortex around *both* the charged particle and the other wormhole mouth. A variant of the argument given earlier shows that carrying the vortex counterclockwise around this mouth changes the wormhole state $|e, b\rangle$ to $|e, hb\rangle$. We thus find that the state of wormhole, particle, and vortex is modified according to

$$\begin{aligned}
 &\frac{1}{\sqrt{n_G}} \sum_{b \in G} D^{(\nu)}(b) |v\rangle_{\text{particle}} \\
 &\otimes |e, b\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}} \\
 &\rightarrow \frac{1}{\sqrt{n_G}} \sum_{b \in G} D^{(\nu)}(h) D^{(\nu)}(b) |v\rangle_{\text{particle}} \\
 &\otimes |e, hb\rangle_{\text{wormhole}} \otimes |h\rangle_{\text{vortex}}, \tag{14}
 \end{aligned}$$

and that the overlap of the initial state with the final state is

$$\begin{aligned} \text{overlap} &= \frac{1}{n_G} \sum_{b, b' \in G} \langle v | D^{(\nu)}(b')^{-1} D^{(\nu)}(hb) | v \rangle \\ &\cdot \langle e, b' | e, hb \rangle \\ &= 1. \end{aligned} \quad (15)$$

Thus the Aharonov–Bohm phase is trivial, and we conclude that the charged particle and mouth are combined together into a singlet state, again as anticipated.

Eq. (1) is a special case of this result. We now understand that if the wormhole mouth initially carries no color charge, that means that the color holonomy associated with traversing the wormhole does not take a definite value. Thus the red quark emerges from the wormhole mouth carrying indefinite color, but with its color perfectly anti-correlated with the color of the mouth. Furthermore, after the (initially) red quark passes through the wormhole, the wormhole state is a superposition of a color octet and color singlet, so that Cheshire charge has been transferred to the wormhole^{#3}.

In summary, we have seen that the β -flux “linked” by a wormhole and the charge of a wormhole mouth cannot simultaneously have definite values. We call this property “wormhole complementarity”. If the β -flux has a definite value, then each wormhole mouth is in an incoherent superposition of charge eigenstates. (The charges of the two mouths are correlated, so that each mouth is described by a mixed density matrix.) There is no Aharonov–Bohm interference when a vortex (or cosmic string) scatters off the mouth. If the charge of each mouth has a definite value, then the wormhole is in a coherent superposition of β -flux eigenstates. Thus, after a colored particle traverses the wormhole, its color is correlated with that of the wormhole mouth from which it emerged.

^{#3} $SU(3)_{\text{color}}$ Cheshire charge has also been discussed recently by Bucher and Goldhaber [15].

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