Improving the precision of quantum metrology using quantum error correction

With Sisi Zhou, Mengzhen Zhang, and Liang Jiang.

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Related work by:

D. Layden and P. Cappellaro

R. Demkowicz-Dobrzański, R. Czajkowski, and P. Sekatski

and many others …

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29 May 2019

Quantum Information Science

Quantum sensing
Improving sensitivity and spatial resolution.

Quantum cryptography
Privacy founded on fundamental laws of quantum physics.

Quantum networking
Distributing quantumness around the world.

Quantum simulation
Probes of exotic quantum many-body phenomena.

Quantum computing
Speeding up solutions to hard problems.

Hardware challenges cut across all these application areas.

Quantum sensing

A *quantum sensor* is (usually) a few-level quantum system that senses something.

High resolution scanning probes of living cells and advanced materials. E.g., NV center = Nitrogen vacancy color center in diamond.

Accelerometers, gyrometers, gravitometers, gravity gradiometers for navigation and surveying. E.g., atom interferometry.

Detection of axions and other dark matter candidates. E.g., superconducting nanowire detectors.

Wanted: Better materials, more precise coherent control, longer coherence times, more efficient readout, compact devices, … and new ideas.

"Next generation" quantum sensing

Higher sensitivity by exploiting squeezing and entanglement. But there is a tradeoff … what enhances sensitivity may also reduce coherence time.

LIGO: Enhanced sensitivity in the current observing run from injecting squeezed light into the dark port of the interferometer.

Quantum radar (a.k.a. quantum illumination). Create entangled photon pair and bounce one photon off a target. Entanglement enhances signal to noise.

What quantum states of multi-qubit sensors provide the best sensing enhancements?

Entangled sensor arrays for geodesy and geophysics: Improved predictions of earthquakes and volcanoes.

Maybe someday: Seeing a city on another planet using a long-baseline network of telescopes performing interferometry using shared quantum entanglement.

Battling noise: Quantum error correction

Measurement precision is limited by *noise*: decoherence and imperfect control of the probe system.

Noise also threatens the scalability of quantum computing. The theory of quantum error correction (QEC) and fault tolerance quantum computing (FTQC) was developed in the mid-1990s to show that scalable quantum computing is possible in principle, under plausible assumptions about the noise.

QEC is a fundamental idea, arguably comparable in importance to the discovery of algorithmic quantum speedups. It can be either active or passive (e.g. topological quantum computing) --- or both.

This talk: How can QEC enhance measurement precision, for a reasonable noise model, and under idealized assumptions about experimental control?

The purpose is to address issues of principle. In particular, how can QEC suppress the noise, without also suppressing the signal?

We'll find a necessary and sufficient condition for achieving Heisenberg-limit scaling of precision with probing time.

Quantum sensing: the standard example(s)

$$H(\omega) = -\frac{\omega}{2} Z, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For example, a spin in a magnetic field.

Our goal: to estimate the *a priori* unknown parameter ω in the Hamiltonian.

Prepare an initial state and then evolve for time t:

$$|\psi(0)\rangle = \left(|0\rangle + |1\rangle\right)/\sqrt{2} \quad \rightarrow \quad |\psi(t)\rangle = \left(|0\rangle + e^{-i\omega t}|1\rangle\right)/\sqrt{2}$$

Measure *X*:

$$\text{Prob}(\pm) = \frac{1}{2}\left(1 \pm \cos(\omega t)\right)$$

Repeat this single-qubit measurement protocol *n* times. How does the precision of our estimate improve with the number of repetitions?

$$\Delta(\omega t) \propto \frac{1}{\sqrt{n}}$$

This is the Standard Quantum Limit (SQL) scaling of the measurement precision with the number of probing qubits. Precision is limited by shot noise.

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Instead of repeating the single-qubit protocol n times, run the protocol once using an n-particle cat state (*Bollinger et al. 1996*).

$$|\psi(0)\rangle = \left(|000\ldots0\rangle + |111\ldots1\rangle\right)/\sqrt{2}$$

$$\rightarrow |\psi(t)\rangle = \left(|000\ldots0\rangle + e^{-in\omega t}|111\ldots1\rangle\right)/\sqrt{2}$$

Measure $\bar{X} = X \otimes X \otimes X \otimes \cdots \otimes X$ (which is equivalent to measuring X for each qubit and computing parity of outcomes).

$$\text{Prob}(\pm) = \frac{1}{2}\left(1 \pm \cos(n\omega t)\right)$$

$$\Delta(\omega t) \propto \frac{1}{n}$$

This is the Heisenberg limit (HL) scaling of measurement precision with the number of probing qubits, so called because it saturates the uncertainty principle applied to the energy ω and the total probing time nt.

But what about noise?

Suppose the probe is subject to dephasing noise in the Z basis with dephasing rate Γ. Then, aside from rotating n times faster than a single qubit, the *n*-qubit cat state also *decoheres n times faster*.

Single qubit with dephasing: $$\text{Prob}(\pm) = \frac{1}{2}\left(1 \pm e^{-\Gamma t}\cos(\omega t)\right)$$

n-qubit cat stat with dephasing: $$\text{Prob}(\pm) = \frac{1}{2}\left(1 \pm e^{-n\Gamma t}\cos(n\omega t)\right)$$

Optimal sensitivity to ω is achieved for $\Gamma t = O(1)$ for single qubit, and for $n\Gamma t = O(1)$ for the *n*-qubit cat state. But that optimal sensitivity is the same in both cases. The entanglement allows us to extract our estimate *n* times faster, but using *n* qubits instead of just one, so the total probing time is the same. Hence entanglement provides no advantage --- how the precision scales with total probing time does not improve, at least for this particular noise scenario (*Huelga et al. 1997*).

Can quantum error correction suppress the noise without suppressing the signal, allowing us to do better?

Example: An idealized "NV center"

To illustrate how coding might help, consider a two-qubit system. An electron spin probes the field, but is subject to bit flip errors. An ancilla nuclear spin does not sense the field, but has a long coherence time. We can apply joint unitary transformations to the two spins.

Electron spin states (frequency = ω): $|0\rangle, \quad |1\rangle.$

Nuclear spin states: $|\uparrow\rangle, \quad |\downarrow\rangle.$

Encode and decode with CNOT. Probe with entangled code state.

$$(|\uparrow\rangle+|\downarrow\rangle)\otimes|0\rangle \xrightarrow{\text{CNOT}} |\uparrow\rangle\otimes|0\rangle+|\downarrow\rangle\otimes|1\rangle$$

$$\xrightarrow{\text{probe}} |\uparrow\rangle\otimes|0\rangle+e^{-i\omega t}|\downarrow\rangle\otimes|1\rangle \xrightarrow{\text{CNOT}} (|\uparrow\rangle+e^{-i\omega t}|\downarrow\rangle)\otimes|0\rangle$$

Check the electron spin after decoding, and if it has flipped, flip it back. Now run this protocol quickly many times in succession, correcting electron bit flips as they occur. The effective time evolution of the nuclear spin is coherent (*Unden et al. 2016*).

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Check the electron spin after decoding, and if it has flipped, flip it back. Now run this protocol quickly many times in succession, correcting electron bit flips as they occur. The effective time evolution of the nuclear spin is coherent (*Unden et al. 2016*) --- no dephasing.

Why does this work?

-- The code space span$\{ |\uparrow\rangle|0\rangle, |\downarrow\rangle|1\rangle\}$ protects against electron spin flips.
-- Evolution governed by Hamiltonian $H(\omega)$ preserves the code space and acts nontrivially within the code space. (The noise is "perpendicular" to signal.)
-- The ancilla is assumed noiseless.
-- We can coherently control the probe and the ancilla.

How far can we generalize this idea?

Assumption 1: Markovian noise

Let's assume that the noise is Markovian, described by a Lindblad master equation.

$$\mathcal{E}_{dt}(\rho) = \sum_a K_a \rho K_a^\dagger, \quad \sum_a K_a^\dagger K_a = I$$

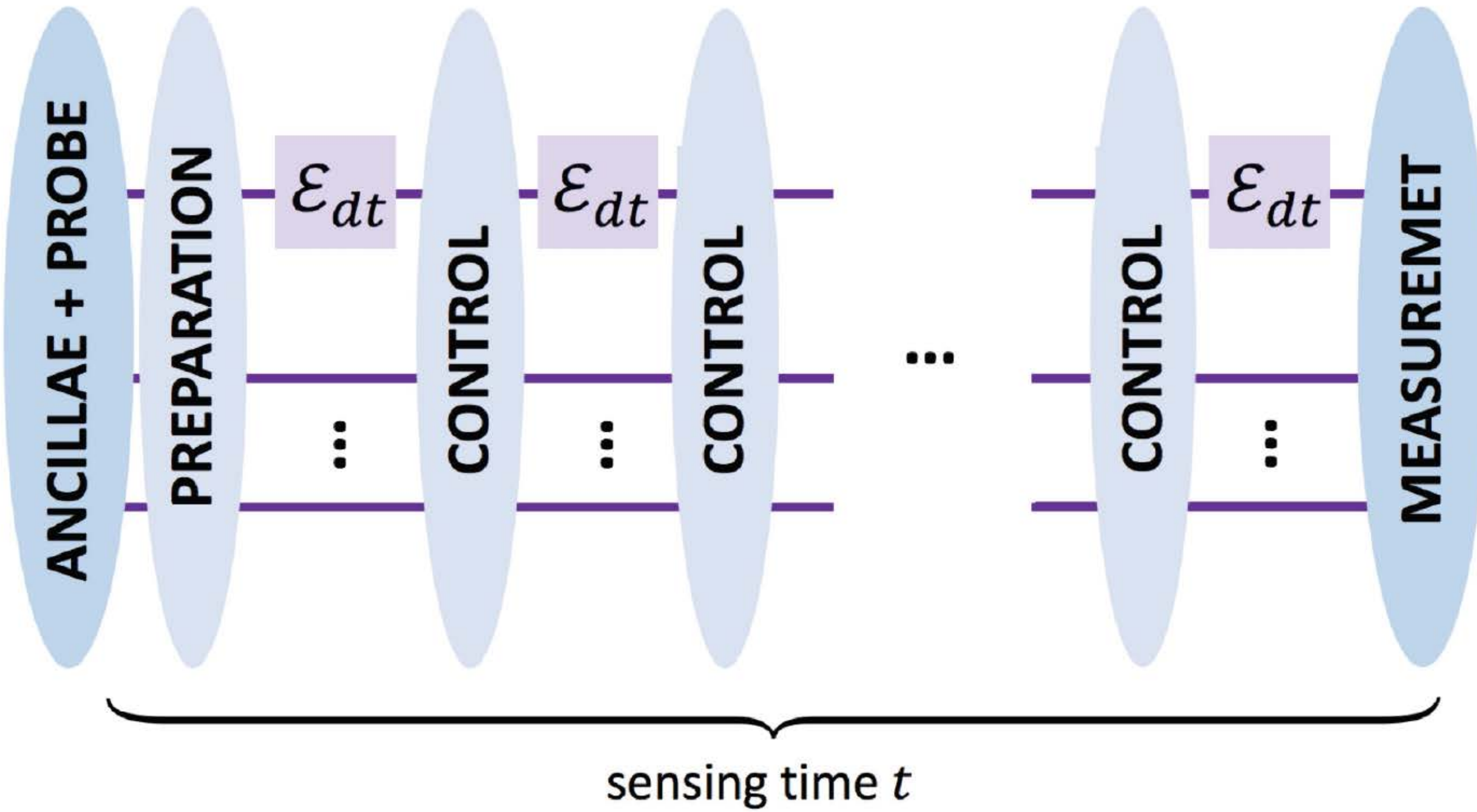
$$K_{a>0} = \sqrt{dt} L_a, \quad K_0 = I + -iHdt - \frac{1}{2} dt \sum_{a>0} L_a^\dagger L_a$$

We'll make us of the gauge freedom in the choice of Kraus operators (freedom to choose a different unraveling of the master equation).

$$K_a^{'} = \sum U_{ab} K_b, \text{ for unitary } U.$$

We'll want a code that can correct the errors described by this master equation.

Assumption 2: Fast and accurate control + noiseless ancilla



Criterion: Heisenberg scaling is achievable if and only if the Hamiltonian is “not in the Lindblad span”

Lindblad span: $$\mathcal{S} = \mathrm{span}\{I, L_a, L_a^\dagger, L_a^\dagger L_b\}$$

Hamiltonian not in Lindblad span (HNLS): $H(\omega) = \notin S$

If the Hamiltonian is Not in the Lindblad Span (HNLS), then a quantum code can be constructed that protects the probe against the noise, and the Heisenberg limit (HL) scaling of precision is achievable (under our assumptions).

If the Hamiltonian is in the Lindblad Span (HNLS fails), then HL is not achievable. The SQL cannot be surpassed.

$$HL \Leftrightarrow HNLS$$

Quantum Fisher Information (QFI)

A probability distribution p_ω depends on the real parameter ω. We want to estimate ω by sampling from p_ω. An unbiased estimator for ω obeys:

$$\mathrm{E}_\omega(\hat{\omega}) = \omega, \quad \partial_\omega \mathrm{E}_\omega(\hat{\omega})|_{\omega=\omega_0} = 1 \to \mathrm{Var}(\hat{\omega}) \geq 1 / J_{\omega_0}$$

(*Cramér-Rao bound*)

Fisher information:

$$J_{\omega_0} = \partial_\omega^2 D\left(p_\omega \,\|\, p_{\omega_0}\right)$$

Relative entropy quantifies distinguishability of distributions.

Quantum Fisher Information, derived from quantum relative entropy, quantifies distinguishability of density operators (*Holevo, Helstrom*, …) and bounds the Fisher information of any measurement to estimate the state (*Braunstein & Caves*).

$$J_{\omega_0} = \partial_\omega^2 D\left(\rho_\omega \,\|\, \rho_{\omega_0}\right), \quad D\left(\rho \,\|\, \sigma\right) = tr\left(\rho \log \rho - \rho \log \sigma\right)$$

If t is the total probing time, then …

$$SQL: J_{\omega_0}(t) \propto t, \quad HL: J_{\omega_0}(t) \propto t^2.$$

Quantum Fisher Information (QFI)

$$J_{\omega_0} = \partial_\omega^2 D\left(\rho_\omega \parallel \rho_{\omega_0}\right), \quad D\left(\rho \parallel \sigma\right) = tr\left(\rho \log \rho - \rho \log \sigma\right)$$

$$SQL: J_{\omega_0}(t) \propto t, \quad HL: J_{\omega_0}(t) \propto t^2.$$

Using ideas from *Fujiwara and Imai 2008*, further developed by *Demkowicz-Dobrzański and Maccone 2014*, we can derive an upper bound on the QFI of the solution to our Markovian master equation, of the form

$$J_{\omega_0}(t) \le At + Bt^2$$

where the coefficients *A* and *B* depend on the chosen gauge for the Kraus operators. Therefore, we are free to choose the gauge to optimize the bound. We find that if HNLS fails, then the gauge can be chosen so that *B*=0. Hence SQL applies. The same result was found independently by *Demkowicz-Dobrzański, Czajkowski, and Sekatski 2018*.

This result confirms the intuition that the Heisenberg Limit is not attainable if the signal cannot be distinguished from the noise.

QEC: The Knill-Laflamme Conditions

A "logical qubit" is encoded using (perhaps many) "physical qubits." We want to protect the logical qubit, with orthonomal basis states $|\bar{0}\rangle$ and $|\bar{1}\rangle$, from a set of possible error operators $\{ E_a \}$. Errors must not destroy the distinguishability of these code basis states.

For protection against bit flips: $E_a|\bar{0}\rangle \perp E_b|\bar{1}\rangle$ (*off-diagonal KL*).

For protection against phase errors: $E_a\left(|\bar{0}\rangle + |\bar{1}\rangle\right) \perp E_b\left(|\bar{0}\rangle - |\bar{1}\rangle\right)$.

In fact, these conditions suffice to ensure the existence of a recovery map.

It follows that: $\langle\bar{0}|E_b^\dagger E_a|\bar{0}\rangle = \langle\bar{1}|E_b^\dagger E_a|\bar{1}\rangle$ (*on-diagonal KL*).

Correctability of the errors means that the operators $E_b^\dagger E_a$ cannot access information about the code state (no leakage of information to the environment).

Code construction I

To construct our code, we use the noisy probe P and the noiseless ancilla A. We may choose the code to be two-dimensional (just one encoded qubit), as this suffices for optimizing our estimator. Error operators act on P, not on A.

To satisfy the *off-diagonal* Knill-Laflamme condition, $E_a|\bar{0}\rangle \perp E_b|\bar{1}\rangle$, we may choose the two code basis states to have orthogonal support on A. (In some cases this won't be necessary, and the ancilla is not needed.)

To satisfy the *on-diagonal* Knill-Laflamme condition, we require that operators in the Lindblad span are unable to distinguish the codewords. That is,

$$0 = \mathrm{tr}_P\left(\rho_{\bar{0}} - \rho_{\bar{1}}\right)O_P, \quad \rho_{\bar{0}} = tr_A\left(|\bar{0}\rangle\langle\bar{0}|\right), \quad \rho_{\bar{1}} = tr_A\left(|\bar{1}\rangle\langle\bar{1}|\right), \quad O_P \in \mathcal{S}.$$

The difference of reduced density operators of the codewords must be orthogonal to the Lindblad span in the Hilbert-Schmidt inner product. This is a nontrivial constraint on the code construction.

There is a further condition: the error-corrected time evolution in the code space induced by the Hamiltonian $H(\omega)$ must be nontrivial; i.e. not proportional to the identity. (Otherwise the code freezes the evolution and there is no signal.)

Code construction II

Here's a way to construct a code that works. Decompose the Hamiltonian into components along and orthogonal to the Lindblad span (in the Hilbert-Schmidt inner product).

$$H(\omega) = H_{\|}(\omega) + H_{\perp}(\omega)$$

We don't care about the component along the Lindblad span --- the KL condition says that it's proportional to the identity when projected onto the code space. Furthermore, the orthogonal component $H_\perp$ must be nonzero if HNLS is satisfied; in addition, $H_\perp$ is traceless, since the identity is in the Lindblad span.

$$H_{\perp} = \sigma_0 - \sigma_1$$

where σ_0 and σ_1 are positive Hermitian operators with equal trace.

Now we can choose the basis states of the code to be normalized purifications of σ_0 and σ_1, which have orthogonal support on the ancilla *A*. Thus the KL conditions are satisfied, and expectation values of the Hamiltonian in the two code basis states have distinct values (nontrivial evolution in the code space).

The effectively noiseless evolution in the code space achieves HL scaling of precision for asymptotically large t. In fact this code achieves optimal precision if $H_\perp$ has rank two (one positive and one negative eigenvalue), but not in general.

Semidefinite program for code optimization

How do we find the best possible code? Consider first the noiseless case (with no ancilla). To get the most precise estimate of ω we should prepare a uniform superposition of energy eigenstates with the largest possible energy difference. That is, we superpose pure states ρ_0 and ρ_1 of the probe that achieve

$$\max \operatorname{tr}\left(\left(\rho_0 - \rho_1\right) H\right) = \lambda_{\max} - \lambda_{\min}$$

This maximum is the difference between the maximal and minimal eigenvalues of *H*.

The same quantity can be computed in an alternative way:

$$\min_{\alpha} \ \| H - \alpha I \ \|_{\text{op}} = \left(\lambda_{\max} - \lambda_{\min}\right) / 2$$

(where the norm is the operator norm). This matching of the maximum and minimum (up to the factor 2) is an instance of *semidefinite programming duality*.

This relationship generalizes to the noisy case. There the maximum is over $\rho_0 - \rho_1$ orthogonal to the Lindblad span (*primal problem*), and the minimum is over *H* shifted by an element of the Lindblad span (*dual problem*). The program can be solved efficiently (in time polynomial in dimension of probe space), and sometimes the dual problem is more convenient to solve.

Example: Kerr nonlinearity and photon loss

Consider a single bosonic mode, subject to loss. The Hamiltonian is quadratic in occupation number *n*, and we want to estimate its coefficient. Suppose the maximum occupation is *n* = 4.

$$\mathcal{S} = \text{span}\{I, a, a^\dagger, a^\dagger a = n\}, \quad H = H_{\parallel} + H_{\perp} = \omega n^2$$

$$I = \text{diag}(1,1,1,1,1), \quad n = \text{diag}(0,1,2,3,4), \quad n^2 = \text{diag}(0,1,4,8,16)$$

$$\mathrm{H}_{\perp} / \omega = n^2 - 4n + 2I = \text{diag}(2,-1,-2,-1,2)$$

$$\mathrm{H}_{\parallel} / \omega = 4n - 2I = \text{diag}(-2,2,6,10,14)$$

$$\max\left(\rho_{\bar{0}} - \rho_{\bar{1}}\right) H_{\perp} = 4\,\omega, \quad \text{achieved by} \quad \left(\rho_{\bar{0}}^* - \rho_{\bar{1}}^*\right) = \text{diag}\left(\frac{1}{2}, 0, -1, 0, \frac{1}{2}\right)$$

In contrast, in the noiseless case, $\lambda_{\max} - \lambda_{\min} = 16\,\omega$ which is four times larger. The noise reduces the precision of our estimate by a factor of 4 if we use the optimal code. Furthermore the ancilla is not needed. We can choose

$$|\,\bar{0}\rangle = \left(|\,0\rangle + |\,4\rangle\right) / \sqrt{2},$$

$$|\,\bar{1}\rangle = |\,2\rangle$$

This is a binomial code, constructed earlier by the *Yale group*. We can measure photon parity to detect the errors.

Example: Spatial filtering for correlated noise

Consider multiple sensors, all detecting a magnetic field in the Z direction, and all subject to dephasing in the Z basis.

In this case, the operators in the Lindblad span and the Hamiltonian are all mutually commuting. If HNLS is satisfied, an optimal code can be constructed without ancillas (Layden and Cappellaro 2018).

A simple case with two sensors: $H \propto Z_1 + Z_2, \quad L \propto Z_1 - Z_2$

Choose the code: $|\bar{0}\rangle = |00\rangle, \quad |\bar{1}\rangle = |11\rangle$

In this case, the code space is actually a decoherence-free subspace (DFS), not affected by the noise at all. Therefore, time evolution preserves the code space, and furthermore, the two code basis states have different energy, so the evolution in the code space is nontrivial. In more general cases, nontrivial recovery is needed, but still no ancilla.

This noise model with one jump operator for two qubits is nongeneric. The same remark applies to HNLS in general --- no HL scaling when noise is full rank.

Extensions of the HNLS criterion

Under our assumptions (Markovian noise, noiseless ancillas, fast and accurate coherent processing) we've shown HL if and only if HNLS.

HNLS is nongeneric but can be approximately satisfied, implying large constant factor improvements in precision (relative to unencoded SQL) achieved by coding.

If Lindblad operators and Hamiltonian are mutually commuting, no ancilla needed to achieve HL with optimal precision if HNLS is satisfied (*Layden, Zhou, Cappellaro, Liang 2019*).

Multiparameter case (*Gorecki, Zhou, Jiang, Demkowicz-Dobrzański 2019*). HL is achievable (for any positive cost matrix) if no linear combination of H_i in Lindblad span. (May not be able to saturate CR bound if measurements incompatible.)

Noisy ancillas. Might use coding and fault tolerance to control the ancilla noise.

What can we say under *realistic* assumptions about accuracy and speed of processing, e.g. when ancilla as well as probe are noisy?

Non-Markovian noise: How to combine dynamical decoupling with coding to achieve optimal precision?