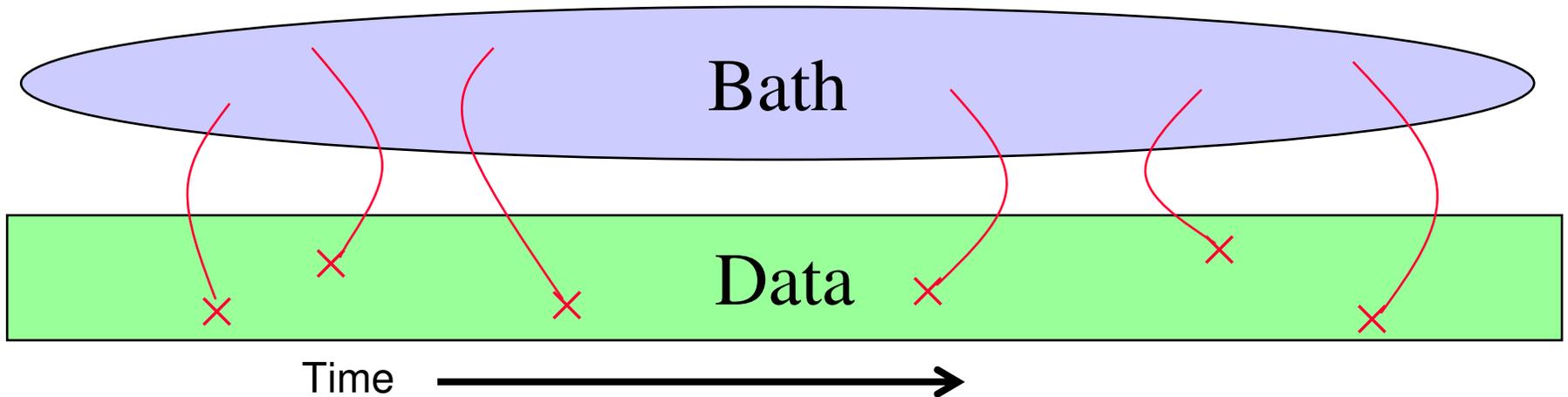
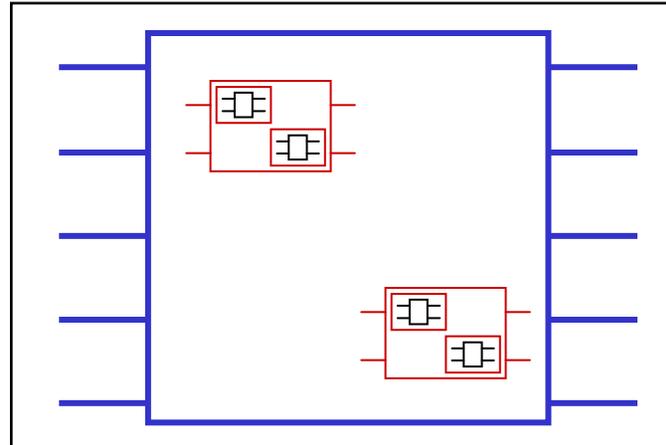


# Fault-tolerant quantum computation versus realistic noise



# Quantum error correction and fault tolerance

Can large-scale quantum computers really be built and operated? Surely there are daunting technical challenges to be overcome. But are there obstacles *in principle* that might prevent us from ever attacking hard computational problems with quantum computers?

What comes to mind in particular is the problem of *errors*. Quantum computers will be far more susceptible to noise than conventional digital computers. A particular challenge is to prevent *decoherence* due to interactions of the computer with the environment. Even aside from decoherence, the unitary quantum gates will not be perfect, and small imperfections will accumulate over time...

Our confidence that large-scale quantum computations will someday be possible has been bolstered by developments in the theory of quantum error correction and fault tolerance.

# Fault-tolerant quantum computation

1. Fault-tolerant quantum computing
2. Quantum accuracy threshold theorem
3. Recent developments
4. Non-Markovian (Gaussian) noise.
5. Conclusions

# Quantum computer: the standard model

- (1) Hilbert space of  $n$  qubits:  $\mathfrak{H} = \mathbb{C}^{2^n}$
- (2) prepare initial state:  $|0\rangle^{\otimes n} = |000\dots 0\rangle$
- (3) execute circuit built from set of universal quantum gates:  $\{U_1, U_2, U_3, \dots, U_{n_G}\}$
- (4) measure in basis  $\{|0\rangle, |1\rangle\}$

The model can be simulated by a classical computer with access to a random number generator. But there is an exponential slowdown, since the simulation involves matrices of exponential size... Thus we believe that quantum model is intrinsically more powerful than the corresponding classical model.

The goal of fault-tolerant quantum computing is to simulate accurately the ideal quantum circuit model using the imperfect noisy gates that can be executed by an actual device (assuming the noise is not too strong).

# Quantum fault tolerance

The goal of fault-tolerant quantum computing is to operate a large-scale quantum computer reliably, even though the components of the computer are noisy.

Reliability can be enhanced by encoding the computer's state in the blocks of a quantum error-correcting code. Each "logical" qubit is stored nonlocally, shared by many physical qubits, and can be protected if the noise is sufficiently weak and also sufficiently weakly correlated in space and time.

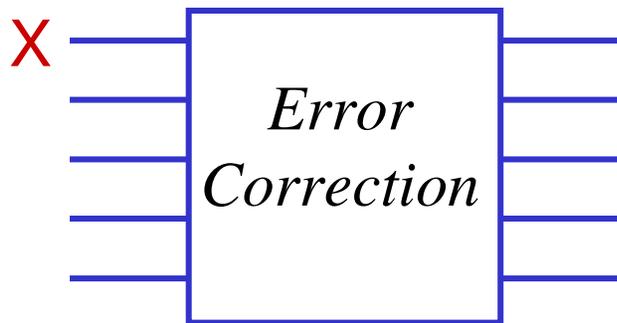
But the encoding and recovery operations are themselves quantum computations. Will quantum coding really protect the computer's state even though the quantum gates we use to recover from error are themselves noisy?

Furthermore, we need to do more than just *store* a quantum state with high fidelity; we also need to *process* the information protected by the code. How do we devise a universal set of *logical* quantum gates that act reliably on the encoded quantum states, even though these logical gates are constructed from noisy physical gates.

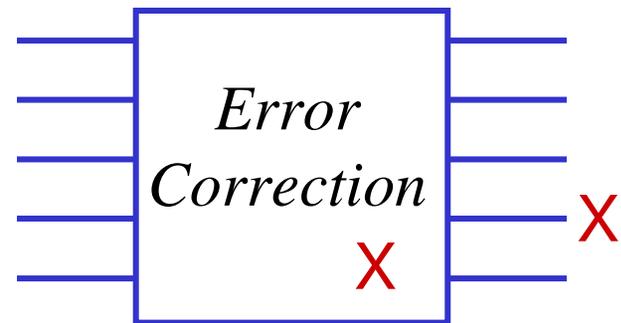
The theory of quantum fault tolerance addresses these questions.

# Fault-tolerant error correction

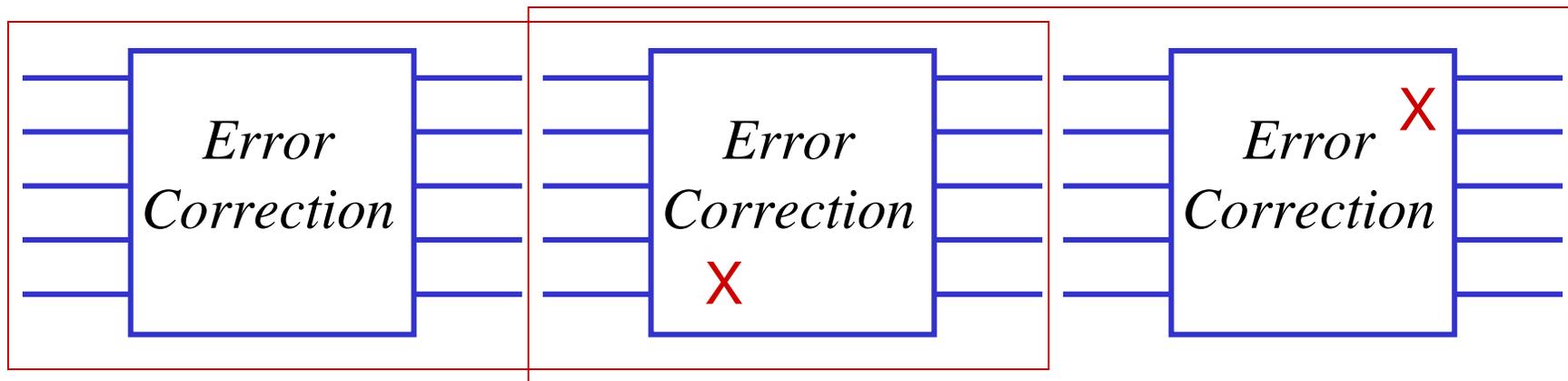
*Fault*: a location in a circuit where a gate or storage error occurs.  
*Error*: a qubit in a block that deviates from the ideal state.



If input has at most one error, and circuit has no faults, output has no errors.



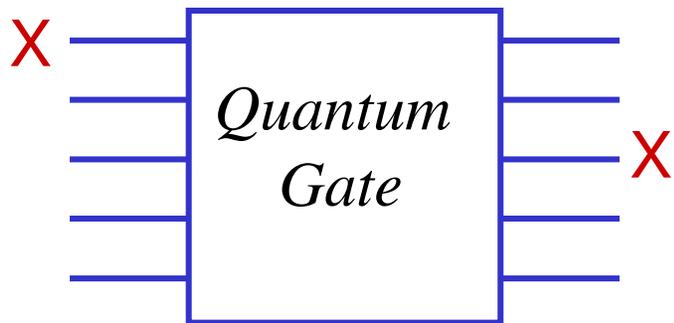
If input has no errors, and circuit has at most one fault, output has at most one error.



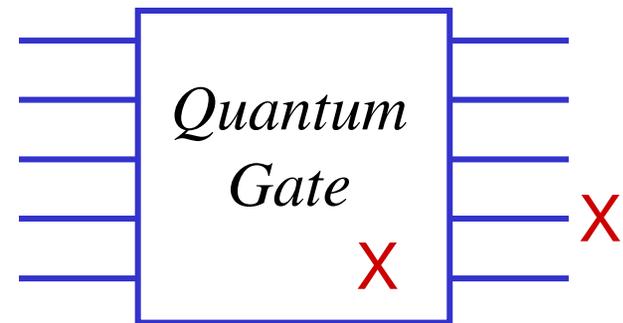
A quantum memory fails only if two faults occur in some “extended rectangle.”

# Fault-tolerant quantum gates

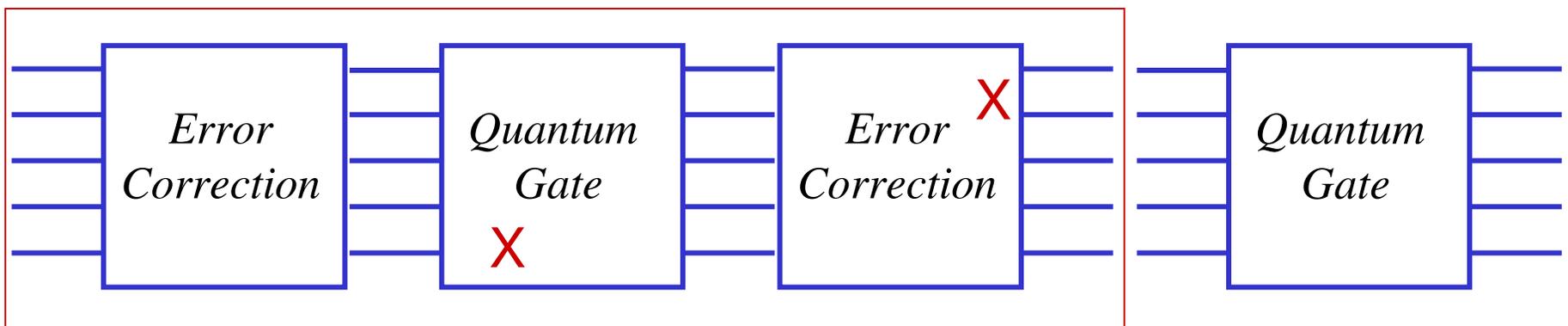
*Fault*: a location in a circuit where a gate or storage error occurs.  
*Error*: a qubit in a block that deviates from the ideal state.



If input has at most one error, and circuit has no faults, output has at most one error in each block.

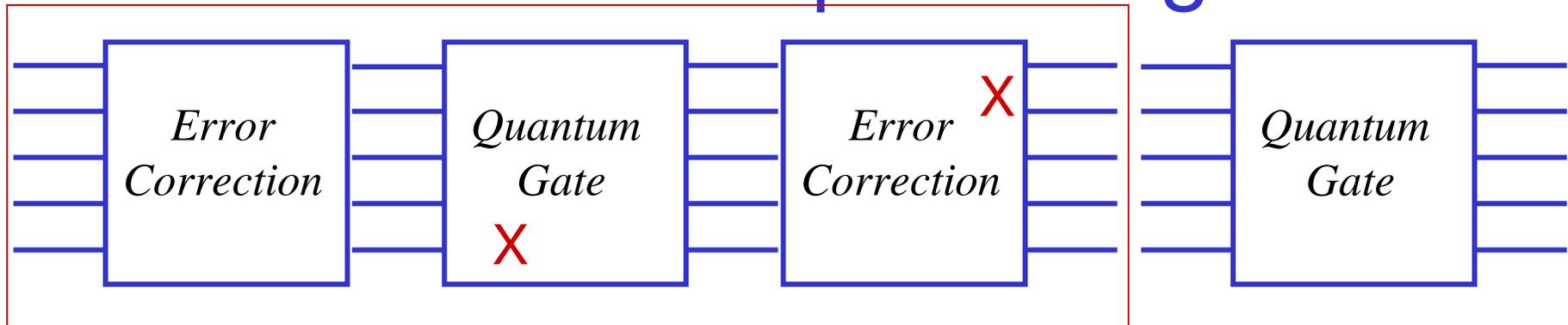


If input has no errors, and circuit has at most one fault, output has at most one error in each block.



Each gate is preceded by an error correction step. The circuit simulation fails only if two faults occur in some “extended rectangle.”

# Fault-tolerant quantum gates



Each gate is followed by an error correction step. The circuit simulation fails only if two faults occur in some “extended rectangle.”

If we simulate an ideal circuit with  $L$  quantum gates, and faults occur independently with probability  $\epsilon$  at each circuit location, then the probability of failure is

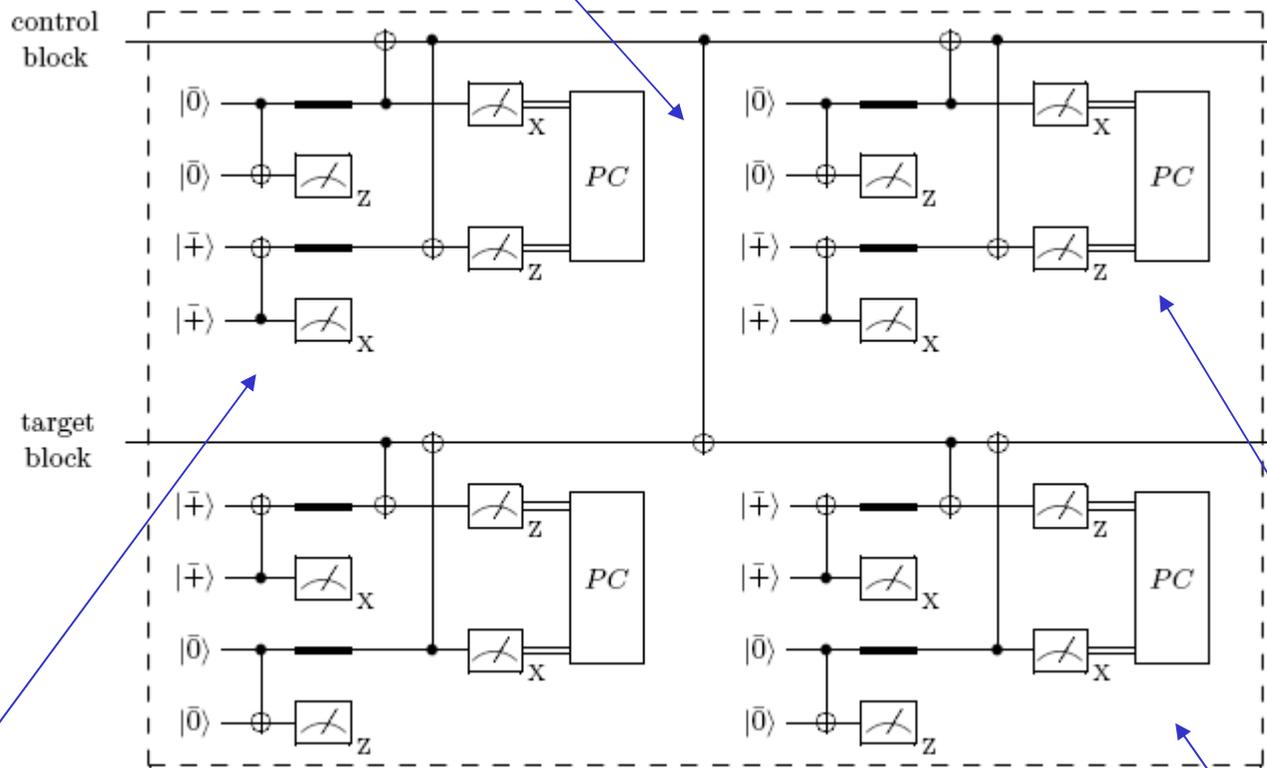
$$P_{\text{fail}} \leq LA_{\text{max}} \epsilon^2$$

where  $A_{\text{max}}$  is an upper bound on the number of pairs of circuit locations in each extended rectangle. Therefore, by using a quantum code that corrects one error and fault-tolerant quantum gates, we can improve the circuit size that can be simulated reliably to  $L=O(\epsilon^{-2})$ , compared to  $L=O(\epsilon^{-1})$  for an unprotected quantum circuit.

# Example: CNOT extended rectangle for a 7-qubit code

each wire represents a 7-qubit block.

logical CNOT gate



error correction steps for control and target blocks

error correction steps for control and target blocks

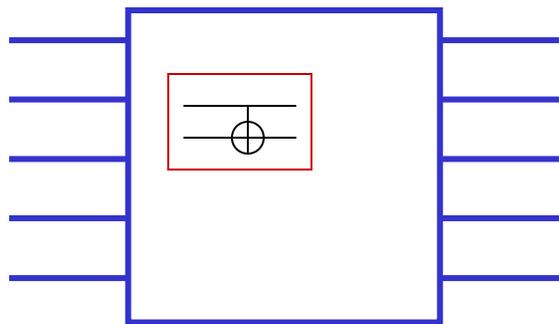
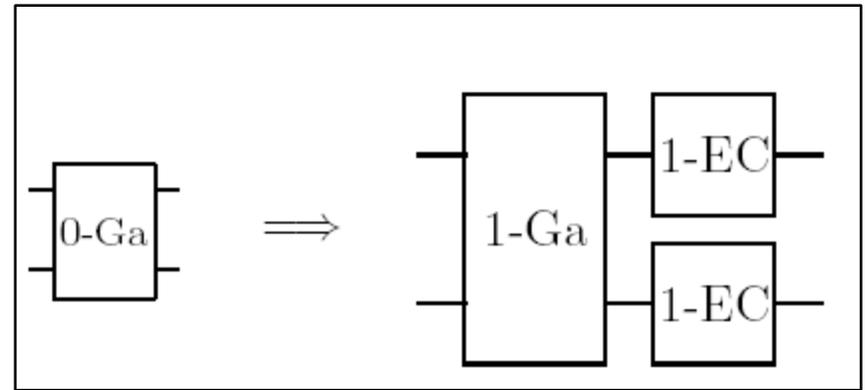
$$4 \times 142 + 7 = 575 \text{ gates}$$

$$165,025 \text{ pairs of gates}$$

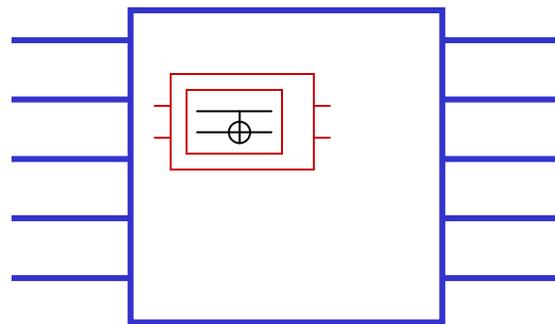
$$35,235 \text{ malignant pairs of gates}$$

# Recursive simulation

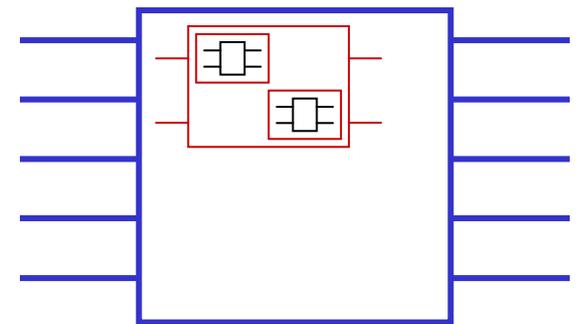
In a fault-tolerant simulation, each (level-0) ideal gate is replaced by a *1-gadget*: a (level-1) gate gadget followed by (level-1) error correction on each output block. In a level- $k$  simulation, this replacement is repeated  $k$  times --- the ideal gate is replaced by a *k-gadget*.



A *1-gadget* is built from quantum gates.



A *2-gadget* is built from 1-gadgets.



A *3-gadgets* is built from 2-gadgets.

The effective noise for the level-1 gadget has a “renormalized” strength:

$$\epsilon^{(1)} \leq \epsilon^2 / \epsilon_0 = \epsilon_0 (\epsilon / \epsilon_0)^2$$

At level  $k$  the renormalized noise strength is:

$$\epsilon^{(k)} < \epsilon_0 (\epsilon / \epsilon_0)^{2^k}$$

# Accuracy Threshold

**Quantum Accuracy Threshold Theorem:** Consider a quantum computer subject to **quasi-independent noise** with strength  $\varepsilon$ . There exists a constant  $\varepsilon_0 > 0$  such that for a fixed  $\varepsilon < \varepsilon_0$  and fixed  $\delta > 0$ , any circuit of size  $L$  can be simulated by a circuit of size  $L^*$  with accuracy greater than  $1 - \delta$ , where, for some constant  $c$ ,

$$L^* = O \left[ L (\log L)^c \right]$$

Aharonov, Ben-Or  
Kitaev  
Laflamme, Knill, Zurek  
Aliferis, Gottesman, Preskill  
Reichardt

*assuming:*

**parallelism, fresh qubits (necessary assumptions)**

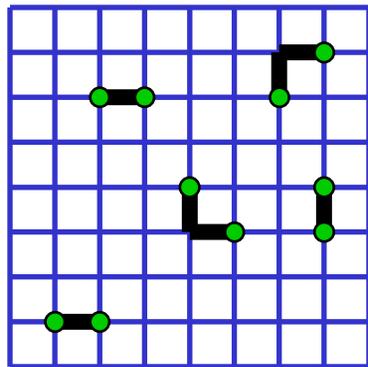
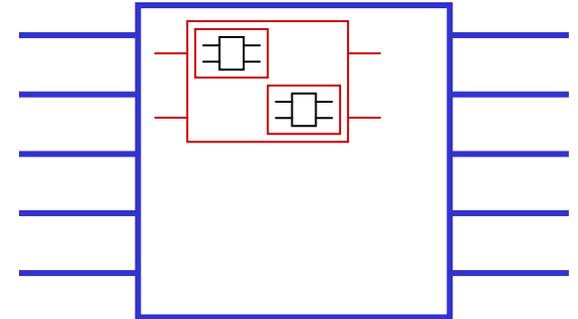
nonlocal gates, fast measurements, fast and accurate classical processing, no leakage (*convenient assumptions*).

The numerical value of the *accuracy threshold*  $\varepsilon_0$  is of practical interest ...

# Accuracy Threshold

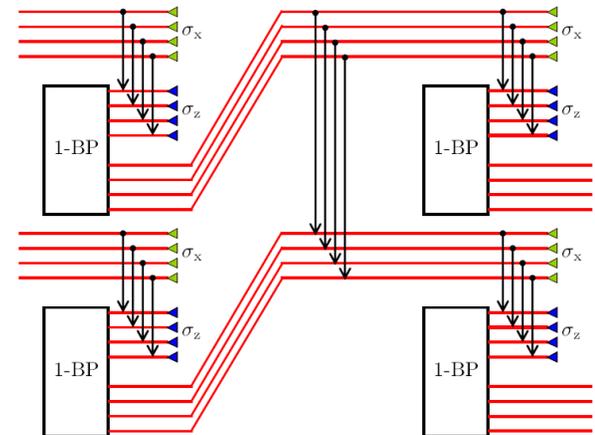
Accuracy threshold theorems have been proved for three types of fault-tolerant schemes:

**Recursive:** hierarchy of gadgets within gadgets, with logical error rate decreasing rapidly with level.



**Topological:** check operators are local on a two-dimensional surface, and detect the *boundary* points of error *chains*. Logical error rate decays exponentially with block's linear size.

**Teleported:** Encoded Bell pairs are prepared recursively, but used only at the top level. The (quantum) depth blowup of the simulation is a constant factor.



# Accuracy Threshold

Estimates of the numerical value of the quantum accuracy threshold estimates have been based on three types of analysis.

**Numerical simulation:** Simulate a stochastic noise model and estimate the probability of gadget failure. Gives the most optimistic threshold estimates, but may not be fully trustworthy, and in any case can be applied only to simple noise models.

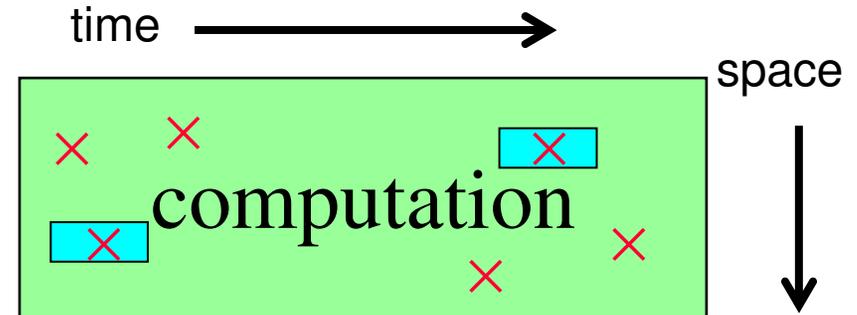
**Rigorous proof:** Prove that quantum computing is scalable for a class of noise models. Gives the most pessimistic threshold estimates, but trustworthy and applicable to noise models not easily amenable to simulation.

**Hybrid methods:** Analytic estimate based on assumptions that some effects can be safely neglected. Typically yields intermediate values for the accuracy threshold.

# Noise models

Two types of noise models are most commonly considered in rigorous estimates of the accuracy threshold.

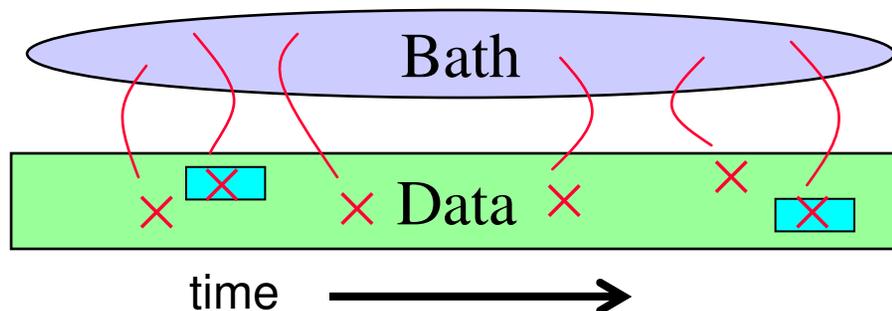
In the **quasi-independent noise model**, “fault paths” are assigned *probabilities*. For any set of  $r$  gates in the circuit, the probability that all  $r$  of the gates have faults is no larger than  $\epsilon^r$ .



The threshold theorem shows that fault-tolerance works for  $\epsilon < \epsilon_0$ . Though not fully realistic, these models provide a useful caricature of noise in actual devices, and can be compared with simulations.

In more realistic **Hamiltonian noise models**, fault paths can add *coherently*. The joint dynamics of the system and “bath” is determined by a Hamiltonian

$$H = H_{System} + H_{Bath} + H_{System-Bath}$$



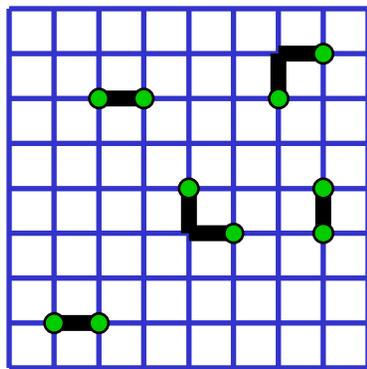
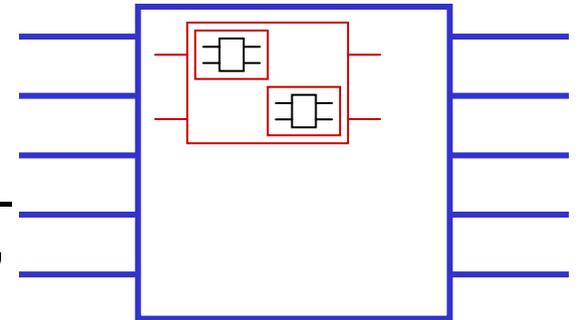
that acts *locally* on the system. Fault tolerance works if the system-bath coupling responsible for the noise is sufficiently weak.

# Accuracy Threshold

Some threshold estimates for stochastic noise:

**Recursive:**  $\epsilon_0 > 1.94 \times 10^{-4}$  proven for quasi-independent noise using “Bacon-Shor codes.”

-- Aliferis, Cross

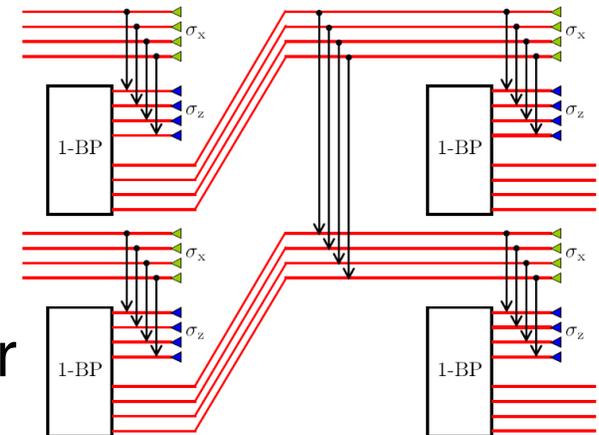


**Topological:**  $\epsilon_0 \sim 7.5 \times 10^{-3}$  estimated for independent depolarizing noise in a **local two-dimensional** measurement-based scheme (combination of numerics and analytic argument).

-- Raussendorf, Harrington, Goyal

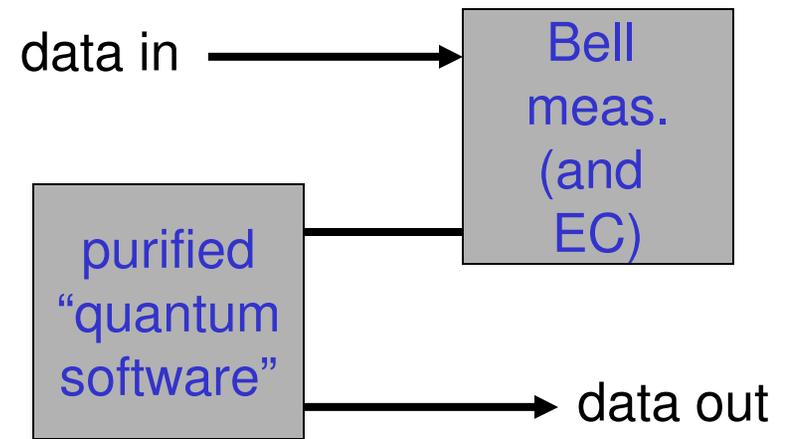
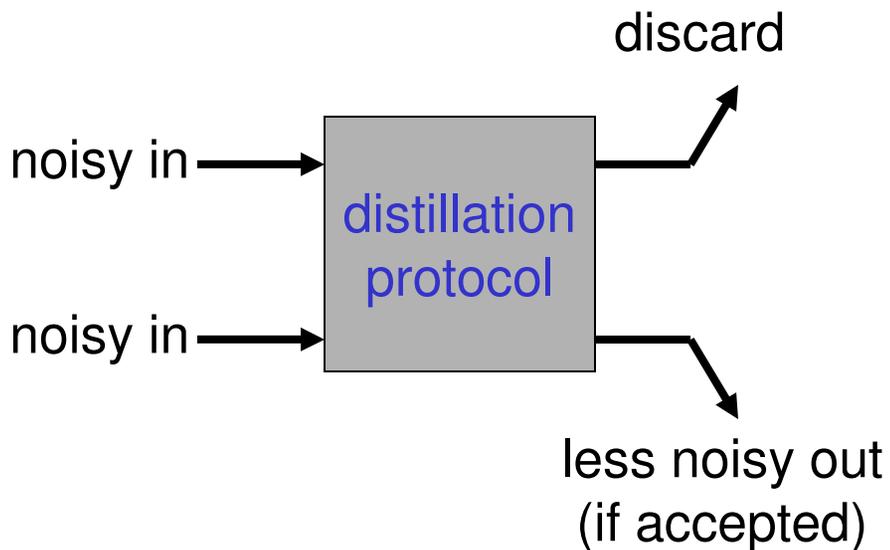
**Teleported (low-overhead version):**

$\epsilon_0 > 6.7 \times 10^{-4}$  proven for quasi-independent noise using concatenated error-detecting codes ( $\epsilon_0 > 1.25 \times 10^{-3}$  for depolarizing noise); simulations indicate  $\epsilon_0 \sim 1 \times 10^{-2}$  for depolarizing noise. -- Knill; Aliferis, Preskill



# Gate teleportation and state distillation

In fault-tolerant schemes, a version of quantum teleportation is used to complete a universal set of protected quantum gates. Suitable “quantum software” is prepared and verified offline, then measurements are performed that transform the incoming data to outgoing data, with a “twist” (an encoded operation) determined by the software.



Reliable software is obtained from noisy software via a multi-round state distillation protocol. In each round (which uses CNOT gates and measurements), there are  $n$  noisy input copies of the software of which  $n-1$  copies are destroyed. The remaining output copy, if accepted, is less noisy than the input copies

Gottesman, Chuang; Knill; Bravyi, Kitaev

# Subsystem codes

Hilbert space  
decomposes as:

$$\mathcal{H} = \bigoplus_s \mathcal{H}_L^{(s)} \otimes \mathcal{H}_G^{(s)}$$

syndrome  $\rightarrow$   $s$       “logical” subsystem      “gauge” subsystem

A subsystem code is really the same thing as a standard quantum code, but where we don't use some of the  $k$  qubits encoded in the code block. These unused qubits are called “gauge qubits” --- we don't care about their quantum state and we don't have to correct their errors.

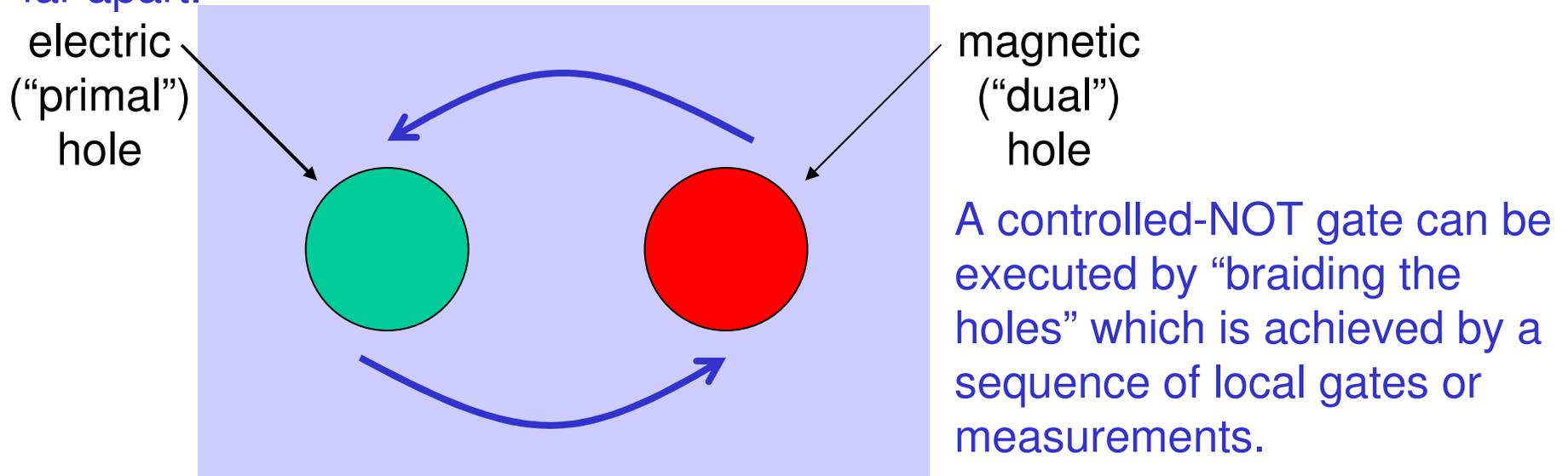
Choosing not to correct the gauge qubits can be surprisingly useful. For one thing, we are free to measure the gauge qubits, and the measurement outcomes can provide useful information about the errors in logical qubits that we really do want to protect.

For example, in the “Bacon-Shor code” it is easier to diagnose the errors by measuring the gauge qubits (measurements of weight-two Pauli operators) than by measuring the standard check operators of the code (measurements of weight-six Pauli operators). This method was used to prove the lower bound on the threshold  $\epsilon_0 > 1.94 \times 10^{-4}$  for a recursive scheme subject to quasi-independent noise.

-- Kribs, Laflamme, Poulin; Bacon; Aliferis, Cross

# Local fault tolerance with 2D topological codes

Qubits are arranged on a two-dimensional lattice with holes in it. Protected qubits are encoded (in either of two complementary bases) by placing “electric” charges inside “primal” holes or “magnetic” charges inside “dual” holes. The quantum information is well protected if the holes are large and far apart.



A controlled-NOT gate can be executed by “braiding the holes” which is achieved by a sequence of local gates or measurements.

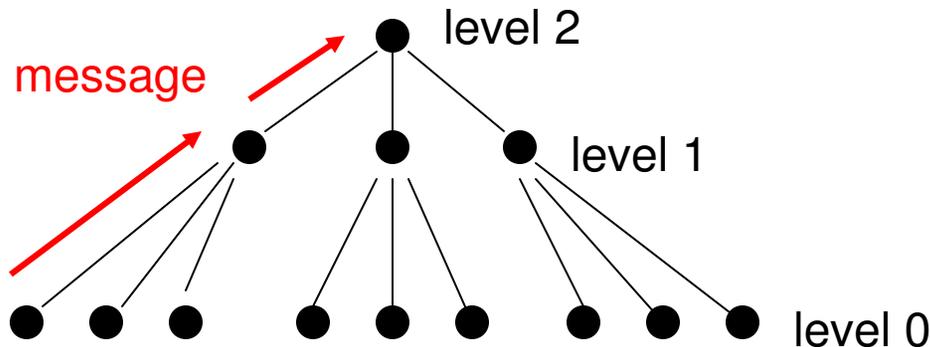
The protected gates and error syndrome extraction can be done with local two-qubit gates or measurements. Numerical studies indicate an upper bound on the threshold for independent depolarizing noise:

$$\epsilon_0 \sim 7.5 \times 10^{-3}$$

Raussendorf, Harrington, Goyal  
Dennis, Kitaev, Landahl, Preskill

# Improved decoding via message passing

The simplest way to decode a concatenated code is “one level at a time”, starting at the lowest level. However, the decoding is more reliable if information about the error syndrome found at lower levels is used to infer the best way to decode at higher levels.



In particular, “message passing” allows a concatenated error-*detecting* code to *correct* errors at higher levels, because an error-detecting code can correct errors that occur at *known* positions in the code block.

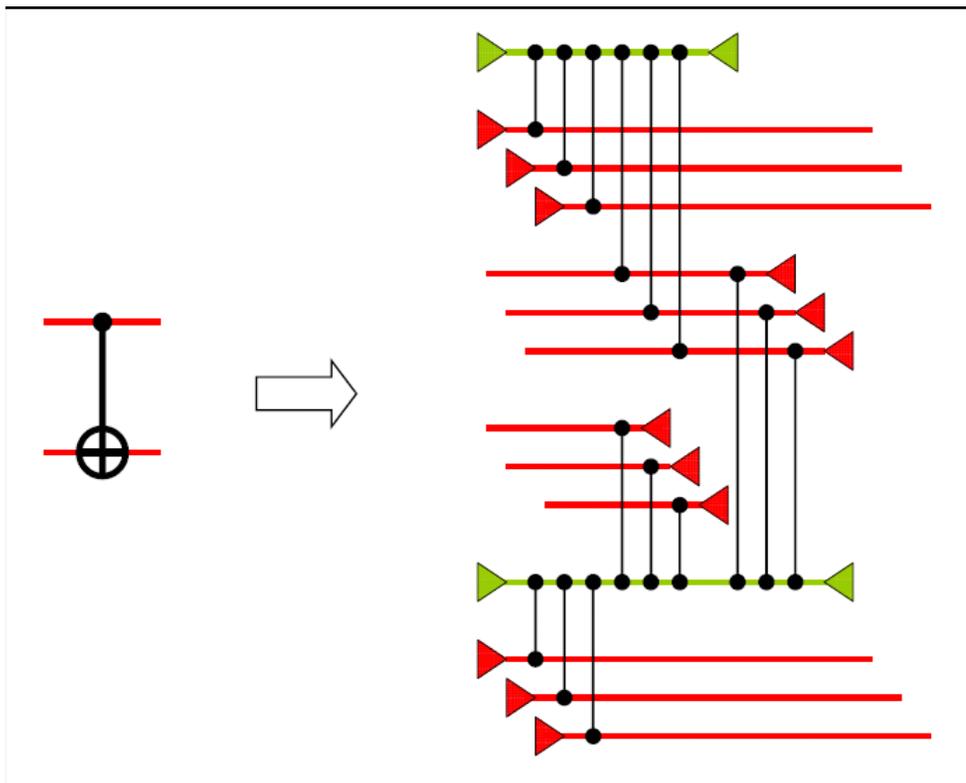
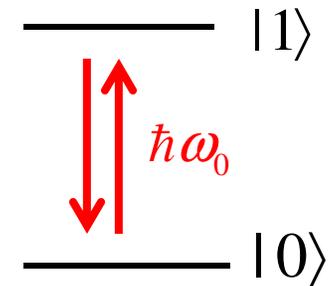
This observation is useful because gadgets for error-detecting codes are simpler than gadgets for error-correcting codes, and hence can tolerate stronger noise. These ideas were applied to prove a lower bound on the accuracy threshold for independent depolarizing noise in the teleported scheme:  $\epsilon_0 > 1.25 \times 10^{-3}$

(simulations indicate  $\epsilon_0 \sim 1 \times 10^{-2}$ )

Poulin; Knill; Aliferis, Preskill

# Fault-tolerant quantum computing versus biased noise

In many physical implementations of quantum gates,  $Z$  noise (dephasing) is stronger than  $X$  noise (relaxation). Dephasing arises from low frequency noise, while relaxation arises from noise with frequency  $\sim \omega_0$ , which is typically much weaker.



Using only controlled-phase two-qubit gates (which are diagonal in the computational basis), plus qubit preparations and measurements, we can do universal fault-tolerant computation with good protection against dephasing. Shown is a circuit that uses these elements to achieve a controlled-NOT gate protected by a phase repetition code. **When the noise is highly biased, the threshold improves by about a factor of 5.**

-- Aliferis, Preskill; Brito, Aliferis, et al.

# Quantum Hardware



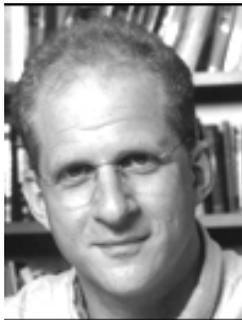
Schoelkopf

Two-level ions in a Paul trap, coupled to “phonons.”

Charge in a Cooper-pair box; fluxons through a superconducting loop.

Electron spin (or charge) in quantum dots.

Cold neutral atoms in optical lattices.



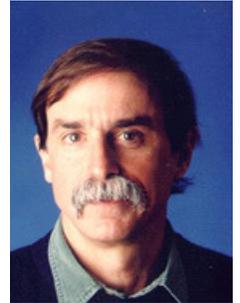
Marcus

Two-level atoms in a high-finesse microcavity, strongly coupled to cavity modes of the electromagnetic field.

Linear optics with efficient single-photon sources and detectors.

Nuclear spins in semiconductors, and in liquid state NMR.

Anyons in fractional quantum Hall systems.



Wineland



Blatt

## Some recently reported error rates

Ion trap – one-qubit gates:

$$4.82 \pm 0.02 \times 10^{-3} \quad [\text{NIST, 2008}]$$

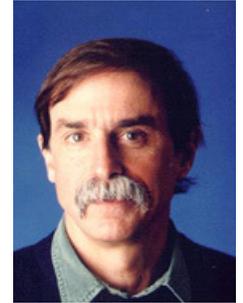
Ion trap – two-qubit gates:

$$\sim 10^{-2} \quad [\text{Innsbruck, 2008}]$$

Superconducting circuits – one-qubit gates

$$1.1 \pm 0.3 \times 10^{-2} \quad [\text{Yale, 2008}]$$

For the single-qubit gates, the error rate was estimated by performing “circuits” of variable size, and observing how the error in the final readout grows with circuit size.



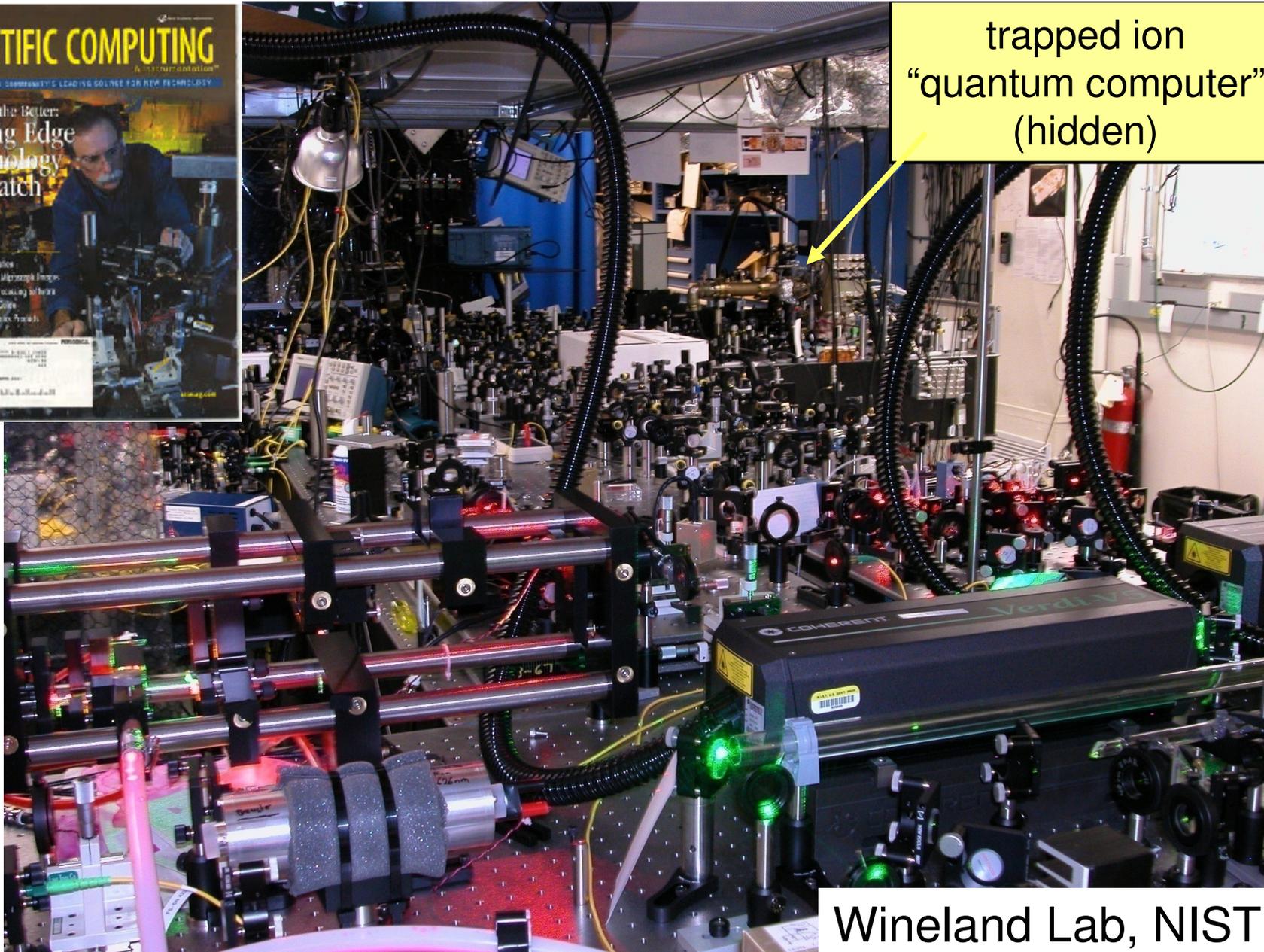
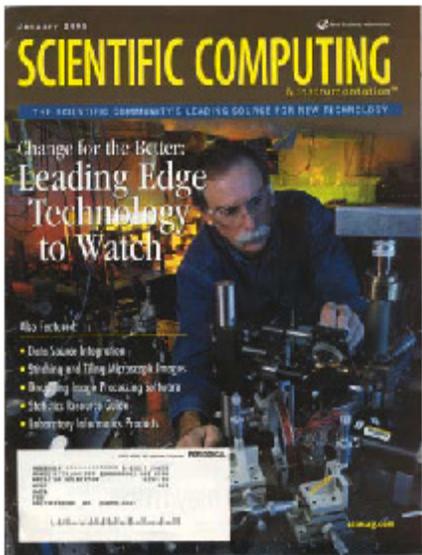
Wineland



Blatt

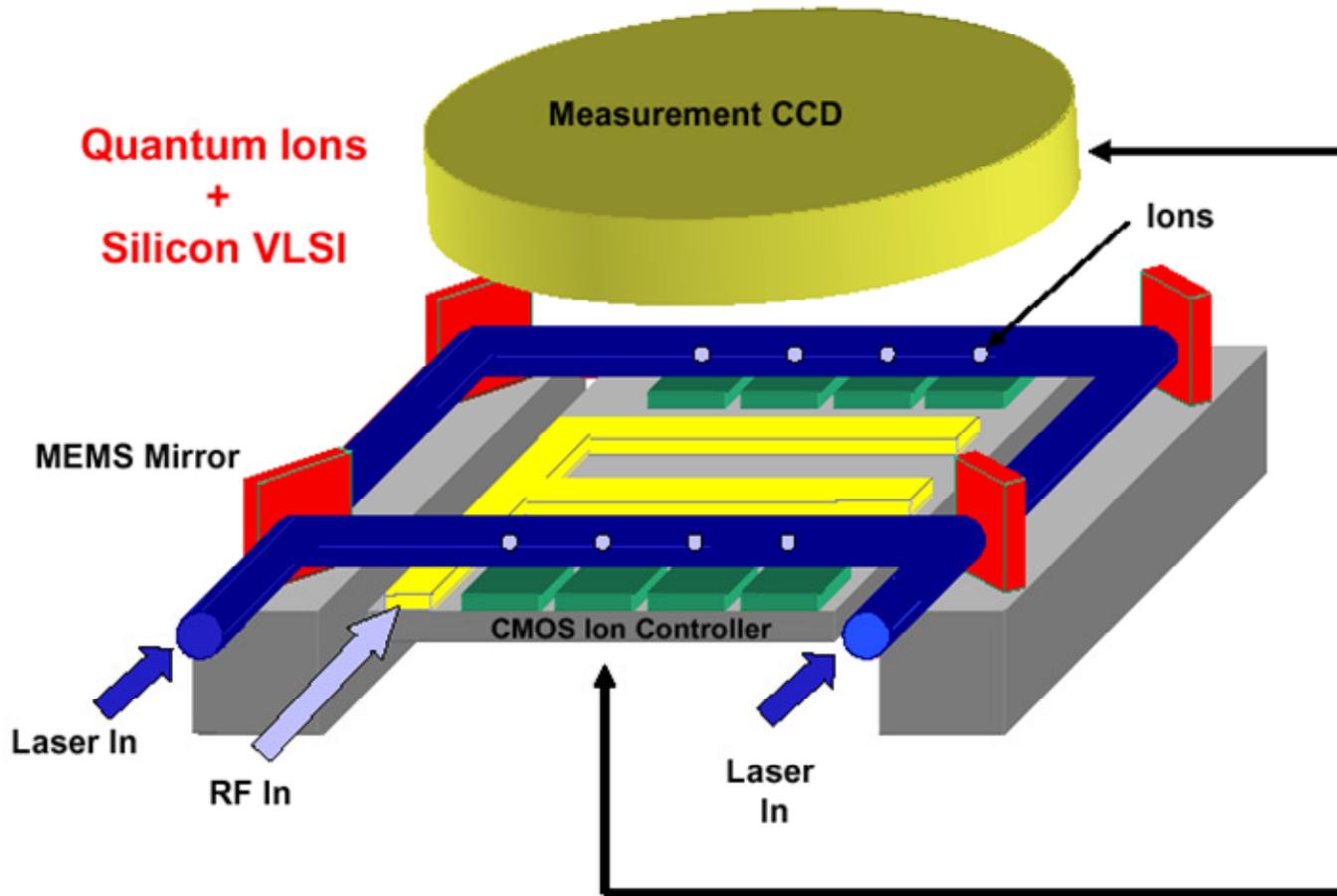


Schoelkopf



Ion trap quantum computer: **The Reality**

# Scalable ion trap quantum computer: **The Dream**



System Compatibility of Quantum & Classical: Spatial Pitch, Clock Speed  
Operating Temperature, Power Dissipation

Dick Slusher  
Bell Labs

# Accuracy Threshold

The quantum accuracy threshold theorem strengthens our confidence that truly scalable quantum computers can be realized in the next few decades.

But might the quest for a quantum computer be frustrated because the noise models assumed by theorists do not accurately describe the noise in actual hardware? We can anticipate that progress toward scalable quantum computing will require an ongoing dialog between experimenters who better understand the limitations of their devices and theorists who propose better ways to overcome these limitations and evaluate the efficacy of these proposals.

Meanwhile, an important task for theorists is to broaden the range of noise models for which useful accuracy threshold theorems can be proven.

# Local non-Markovian noise

Terhal, Burkard  
Aliferis, Gottesman, Preskill  
Aharonov, Kitaev, Preskill

From a physics perspective, it is natural to formulate the noise model in terms of a Hamiltonian that couples the system to the environment.

Non-Markovian noise with a *nonlocal* bath.

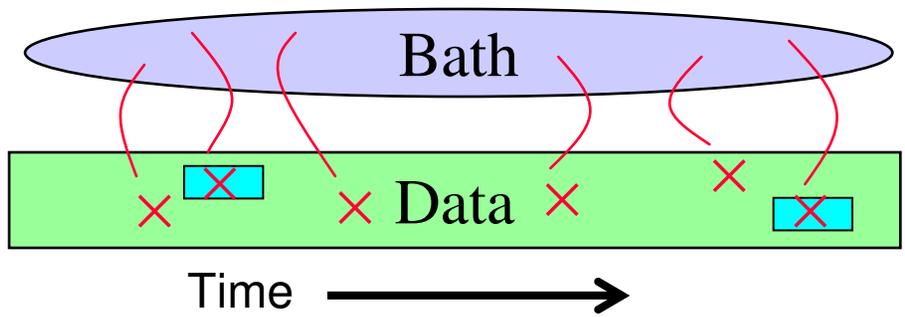
$$H = H_{System} + H_{Bath} + H_{System-Bath}$$

where 
$$H_{System-Bath} = \sum_{\text{terms } a \text{ acting locally on the system}} H_{System-Bath}^{(a)}$$

Then

$$U_{SB} = \sum \text{“Fault Paths”}$$

For *local (coherent) noise* with strength  $\epsilon$ , the norm of the sum of all fault paths such that  $r$  specified gates are faulty is at most  $\epsilon^r$ .



$$\epsilon = \max \left\| H_{System-Bath}^{(a)} \right\| t_0$$

over all times and locations

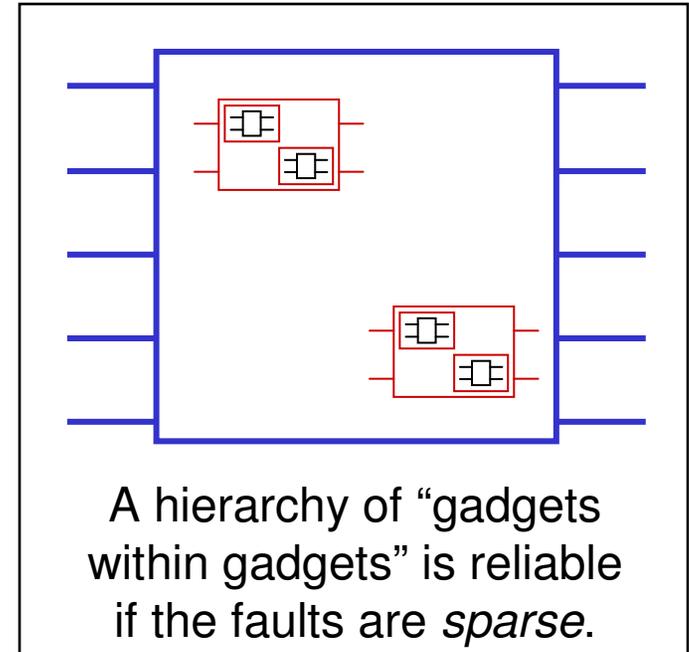
time to execute a gate

# Local non-Markovian noise

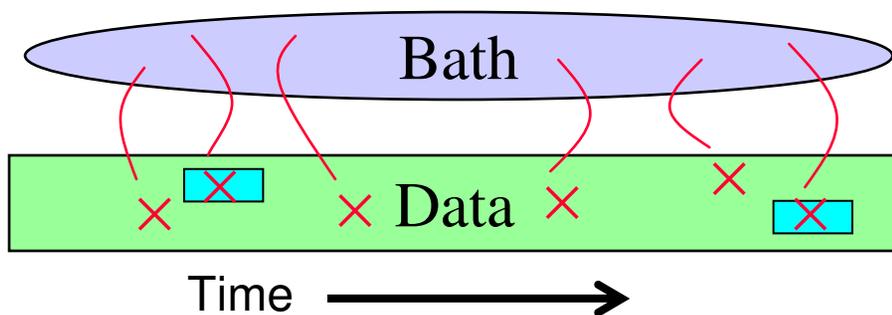
Non-Markovian noise with a *nonlocal* bath.

$$H = H_{System} + H_{Bath} + H_{System-Bath}$$

We can find a rigorous upper bound on the norm of the sum of all “bad” diagrams (such that the faults are *not* sparsely distributed in spacetime). Fault-tolerant quantum computation is effective if the noise strength  $\epsilon$  is small enough, e.g.,  $\epsilon < 10^{-4}$ .



Quantum error correction works as long as the coupling of the system to the bath is *local* (only a few system qubits are jointly coupled to the bath) and *weak* (sum of terms, each with a small norm). Arbitrary (nonlocal) couplings among the bath degrees of freedom are allowed.

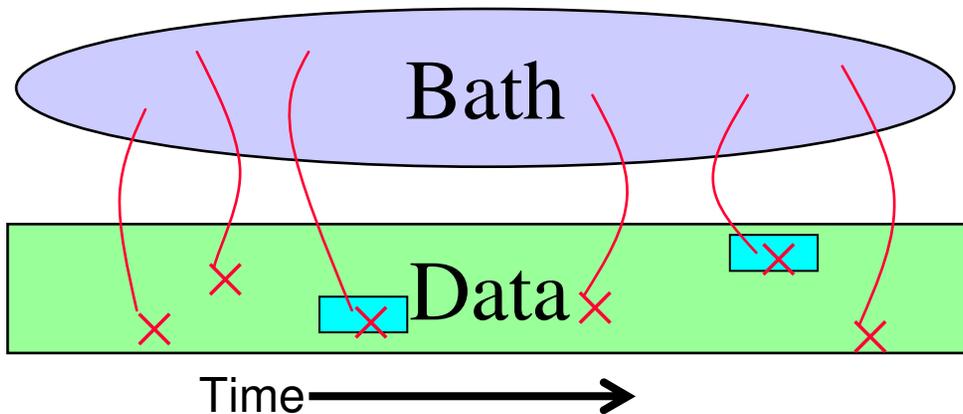


$$\epsilon = \max \left\| H_{System-Bath}^{(a)} \right\| t_0 < \epsilon_0$$

over all times and locations

time to execute a gate

## Local non-Markovian noise

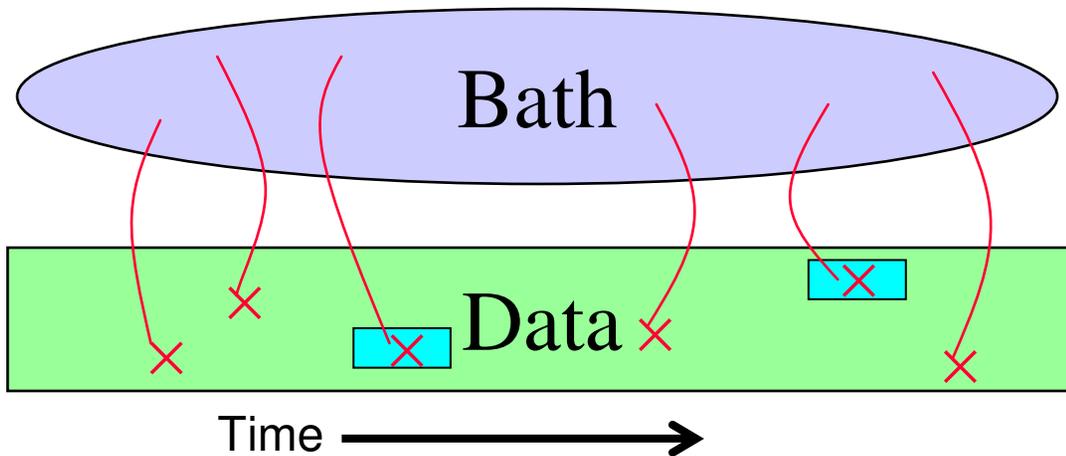


$$\max \left\| H_{\text{System-Bath}}^{(a)} \right\| t_0 < \varepsilon_0 \approx 10^{-4}$$

However, expressing the threshold condition in terms of the norm of the system-bath coupling has disadvantages.

- This condition seems discouraging because it requires an *amplitude* rather than a *probability* (square of an amplitude) to be small. (We pessimistically allow the bad fault paths to have a common phase and interfere constructively.) Under a plausible “phase randomization” hypothesis this estimate could be improved, but it is not so obvious what further assumptions we should make about the noise model to justify a rigorous argument that incorporates “phase randomization”.
- The norm of the system-bath Hamiltonian is not directly measurable in experiments, and in fact for some noise models (e.g. coupling to a bath of harmonic oscillators) the norm is infinite. It would be more natural, and more broadly applicable, if we could express the threshold condition in terms of the *correlation functions* of the noise, which are experimentally accessible.

# Local non-Markovian noise



$$\left\| H_{\text{System-Bath}}^{(a)} \right\| t_0 < \varepsilon_0 \approx 10^{-4}$$

However, expressing the threshold condition in terms of the norm of the system-bath coupling has disadvantages.

If we are willing to make further assumptions about the noise model, we *can* formulate a threshold condition in terms of the noise correlation functions (i.e. the power spectrum of the bath fluctuations).

We will consider the case where each qubit couples to an (initially) thermal bath of harmonic oscillators. Our task is to estimate in this model the the norm squared of the bad part of the system-bath wave function:

At least one insertion of perturbation at each of  $r$  marked locations

$$\left\| \psi_{SB}^{bad} \right\|^2 = \langle \psi_{SB}^0 | U_{SB}^{bad \dagger} U_{SB}^{bad} | \psi_{SB}^0 \rangle \leq \varepsilon^{2r}$$

purification of bath state (if mixed)

# Gaussian noise model

Ng, Preskill

In the *Gaussian noise model*, each system qubit couples to a bath of harmonic oscillators:

$$H = H_S + H_B + H_{SB} \quad H_B = \sum_k \omega_k a_k^\dagger a_k \quad (\text{uncoupled oscillators})$$

$$H_{SB}(t) = \sum_x \sum_\alpha \sigma_\alpha(x) \otimes \tilde{\phi}_\alpha(x, t)$$

( $x$  labels qubit position,  $\tilde{\phi}$  is a Hermitian bath operator,  $\sigma$  is Pauli operator acting on the system qubit,  $g$  is a coupling constant.)

$$\tilde{\phi}_\alpha(x, t) = \sum_k g_{k,\alpha}(x, t) a_k + g_{k,\alpha}(x, t)^* a_k^\dagger$$

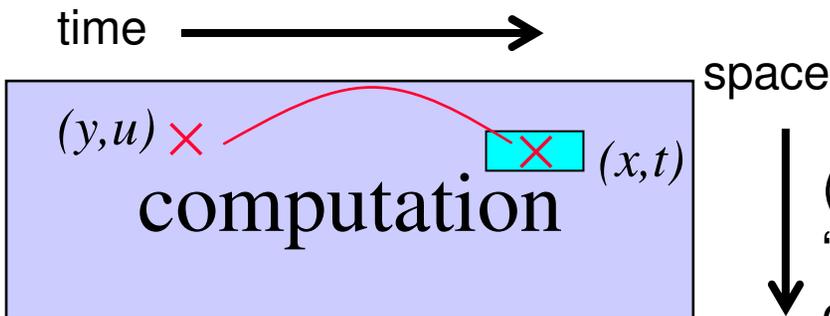
If the initial state of the bath is Gaussian (e.g., thermal), the provable threshold condition can be expressed in terms of the two-point correlation function of the bath variables:

$$\mathcal{E} = \max \left( C \int_{\square} dt \int_{all} du \sum_{y,\alpha,\beta} \left| \langle \phi_\alpha(x, t) \phi_\beta(y, u) \rangle_B \right| \right)^{1/2} < \mathcal{E}_0$$

Here  $\phi$  is the “interaction picture” bath variable

$$\phi_\alpha(x, t) \equiv e^{iH_B t} \tilde{\phi}_\alpha(x, t) e^{-iH_B t}$$

$$= \sum_k g_{k,\alpha}(x, t) a_k e^{-i\omega_k t} + g_{k,\alpha}(x, t)^* a_k^\dagger e^{i\omega_k t}$$



$(x, t)$  is integrated over one circuit “location”, while  $(y, u)$  is integrated over all of spacetime.

# Gaussian noise model

We say that the noise is *Gaussian* because the fluctuations of the bath obey Gaussian statistics: all correlation functions are determined by the two-point correlators. For a shorthand, denote  $\langle \phi_{\alpha_1}(x_1, t_1) \phi_{\alpha_2}(x_2, t_2) \rangle_B \equiv \Delta(1, 2)$

Then

$$\langle \phi(1)\phi(2)\phi(3)\phi(4) \rangle_B \equiv \Delta(1, 2)\Delta(3, 4) + \Delta(1, 3)\Delta(2, 4) + \Delta(1, 4)\Delta(2, 3)$$

(a sum of “contractions”). Applies not just to vacuum expectation value, but also to expectation value in a thermal state of the bath (“Wick’s theorem”).

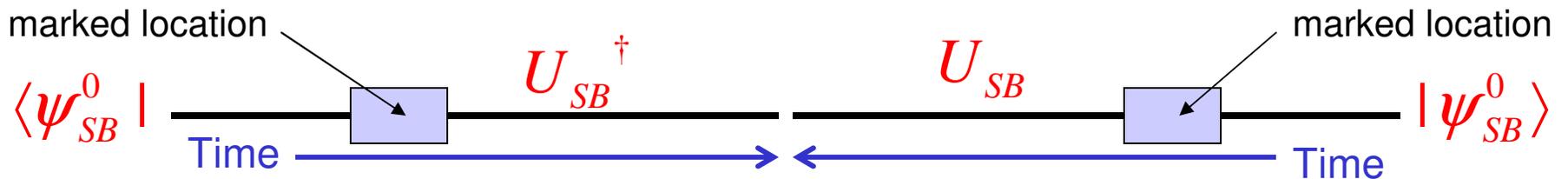
$$\langle \phi(1)\phi(2)\phi(3)\phi(4) \rangle_B = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \quad | \quad | \quad | \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \quad | \quad | \quad | \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ | \quad | \quad | \quad | \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}$$

Similarly, the 2n-point correlation function can be expressed as a product of two-point correlators, summed over all possible pairwise contractions.

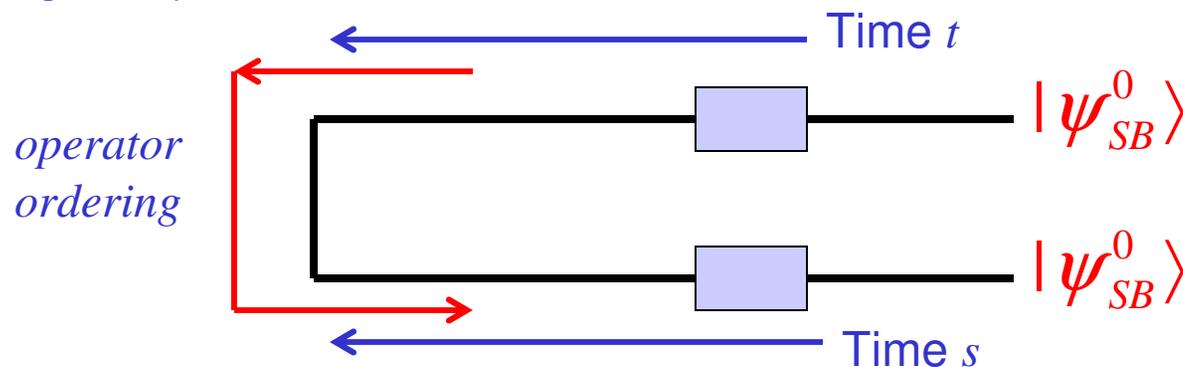
$$\langle \phi(1)\phi(2)\cdots\phi(2n) \rangle_B = \sum_{\text{contractions}} \Delta(i_1, j_1)\Delta(i_2, j_2)\cdots\Delta(i_n, j_n)$$

# Gaussian noise model

Now we consider the case where  $r = 1$  location(s) in the quantum circuit is “bad”; i.e., has at least one insertion of the perturbation. We are to sum all the “bad” contributions to the norm squared of the (pure) state of system and bath.



It is convenient to bend this picture into a hairpin shape (“Schwinger-Keldysh diagram”)



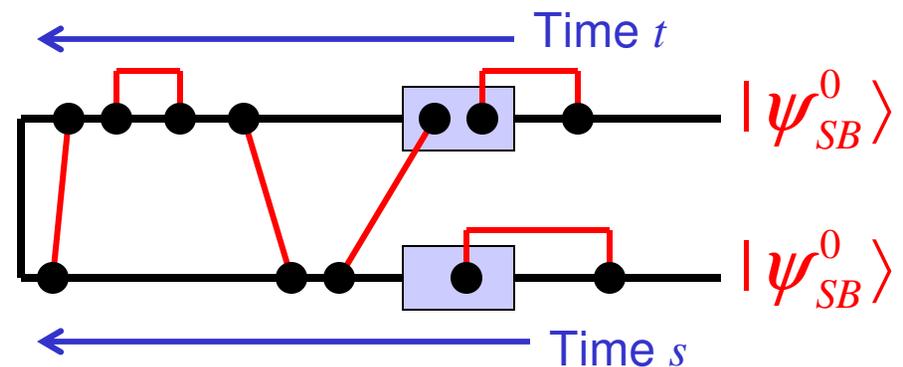
Time increases to the left on both branches, but “time-ordered” operators on the “upper branch” act “before” “anti-time-ordered” operators on the “lower branch”.

# Gaussian noise model

Now we consider expanding the time evolution operator  $U_{SB}$  in powers of the perturbation  $H_{SB}$ , summed to all orders. For a fixed term in this expansion, the system and the bath are uncoupled in between insertions of  $H_{SB}$ : the system evolves ideally between insertions, as determined by  $H_S$ , and the bath evolves as determined by  $H_B$  (“interaction picture”).

Thus tracing out the bath generates the expectation value of a product of bath fields in the interaction picture, which can be evaluated using Wick’s theorem (i.e., using the Gaussian statistics of the bath fluctuations). This is accompanied by the expectation value in the system’s initial state of a product of interaction picture operators acting on the system qubits.

We are to sum up all the diagrams with at least one insertion of the perturbation inside the marked location on each branch of the Keldysh diagram.

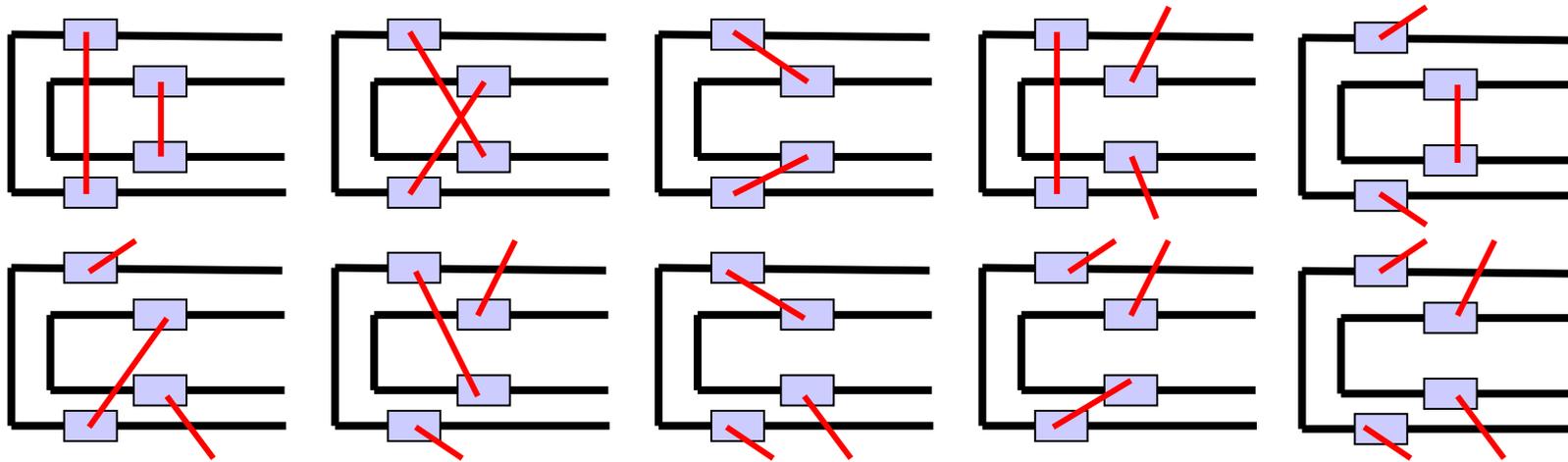


This sum is the norm squared of the bad part of the system-bath state:

$$\langle \psi_{SB}^0 | U_{SB}^{bad \dagger} U_{SB}^{bad} | \psi_{SB}^0 \rangle$$

# Gaussian noise model

When there are  $r$  marked locations in the circuit, we get a bound on norm squared of the bad part by summing over all ways to contract the marked locations, either with one another or with external locations (shown for  $r=2$ ).



Using the same methods as in AKP05, we can bound the sum of the absolute values of all the diagrams, finding:  $\langle \psi_{SB}^0 | U_{SB}^{bad\dagger} U_{SB}^{bad} | \psi_{SB}^0 \rangle \leq \epsilon^{2r}$   
 where:

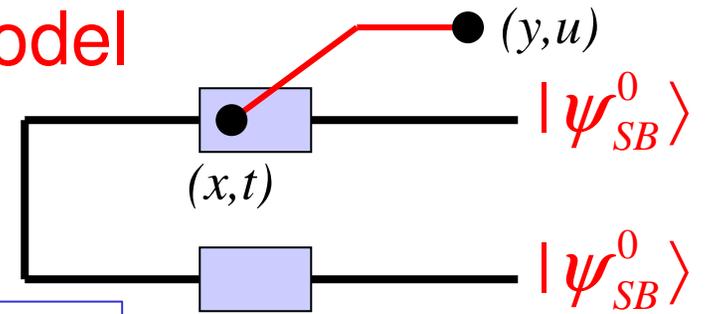
$$\epsilon^2 = \max_{\square} \left( C \int ds \int du \sum_y \sum_{\alpha, \gamma} | \langle \phi_{\alpha}(x, s) \phi_{\beta}(y, u) \rangle_B | \right)$$

and  $C \approx 2e = (2.34)^2$

In this noise model, fault-tolerant quantum computing works if  $\epsilon$  is small enough (e.g. smaller than  $10^{-4}$ ).

## Gaussian noise model

In this noise model, fault-tolerant quantum computing works if  $\varepsilon$  is small enough (e.g. smaller than  $10^{-4}$ ).



$$\varepsilon = \max_{\square} \left( C \int_{all} dt \int_{all} du \sum_{y, \alpha, \beta} \left| \langle \phi_{\alpha}(x, t) \phi_{\beta}(y, u) \rangle_B \right| \right)^{1/2} < \varepsilon_0$$

If correlations are critical (decay like a power), then this expression converges provided

$$\int_{all} du \sum_y |\Delta(x, t; y, u)| < \infty$$

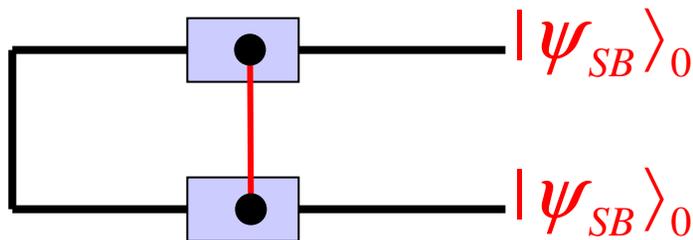
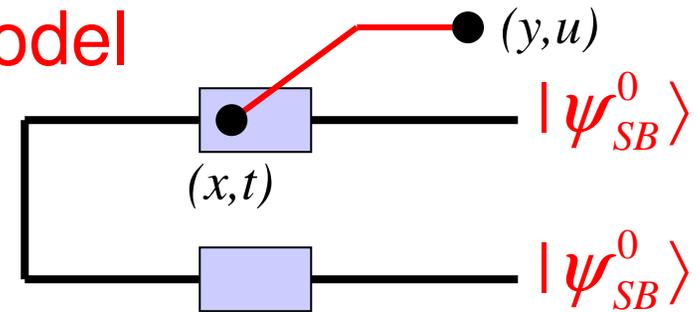
where  $|\Delta(x, t; y, u)| = \sum_{\alpha, \beta} \left| \langle \phi_{\alpha}(x, t) \phi_{\beta}(y, u) \rangle_B \right|$

or  $\int_{all} dt \int_{all} d^D x \frac{1}{(x^2 + t^{2/z})^{\delta}} < \infty$  i.e.  $D + z < 2\delta$

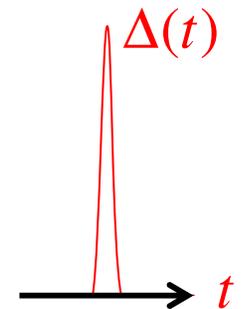
(Here we assume the qubits are uniformly distributed in  $D$ -dimensional space,  $\delta$  is the scaling dimension of the bath field, and  $z$  is the dynamical critical exponent.) Cf, **Novais et al.**

# Gaussian noise model

$$\varepsilon = \left[ C \int_{\square} dt \int_{all} du \sum_y |\Delta(x,t; y,u)| \right]^{1/2}$$



In the Markovian limit, the correlator is a delta function in time, with support at vanishing time difference:



Thus:  $\varepsilon = (\Gamma t_0)^{1/2}$

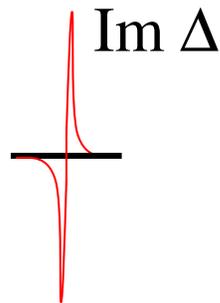
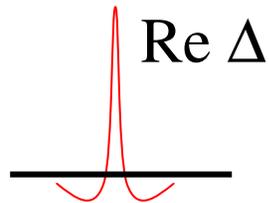
where  $\Gamma$  is an error *rate*, and  $t_0$  is the time to execute a gate. In the Markovian case, fault paths really do decohere, and errors can be assigned probabilities rather than amplitudes. But our argument is not clever enough to exploit this property, and hence our threshold condition requires the error amplitude to be small, rather than the square of the amplitude.

This result applies to “high temperature” Ohmic noise, which has a flat power spectrum up to a cutoff frequency (i.e. the inverse width of the peak). The norm condition, on the other hand, requires the *height* of the peak in the correlator to be small, a quantity that depends on the frequency cutoff.

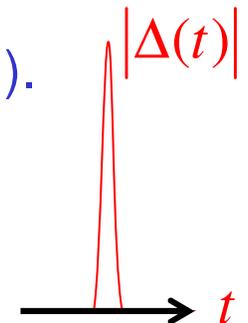
# Gaussian noise model

In the case of zero-temperature Ohmic noise,

$$\tilde{\Delta}(\omega) = 2\pi A\omega e^{-\omega\tau_c} \quad \text{and} \quad \Delta(t_1 - t_2) = \frac{-A}{\left[ (t_1 - t_2) - i\tau_c \right]^2}$$



Both the real and the imaginary part of the correlator wiggle, and therefore the integral of the correlator has only a logarithmic sensitivity to the cutoff (cf. Novais et al.).



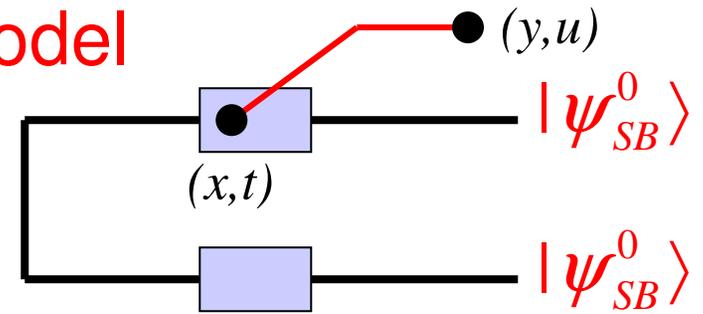
However, unfortunately when we take the absolute value of the correlator, we lose the benefit of the wiggles, and the cutoff dependence is stronger:

$$\varepsilon^2 \approx \int_0^{t_0} dt_1 \int_{-\infty}^{\infty} dt_2 |\Delta(t_1 - t_2)| = \int_0^{t_0} dt_1 \int_{-\infty}^{\infty} dt_2 A / \left[ (t_1 - t_2)^2 + \tau_c^2 \right] = \pi A (t_0 / \tau_c)$$

The height of the peak is  $\tau_c^{-2}$  and its width is  $\tau_c$ . By integrating, we improve the value of  $\varepsilon$  relative to our original norm condition by a factor  $(\tau_c / t_0)^{1/2}$ . Still, rather strong sensitivity to the cutoff remains (in the zero-temperature Ohmic case).

## Gaussian noise model

Thus in some cases (like high-temperature Ohmic noise) our new threshold condition for Gaussian noise has no artificial sensitivity to very-high-frequency fluctuations of the bath, while in other cases (like zero-temperature Ohmic noise) sensitivity to the cutoff remains, yet is improved compared to the norm condition of Terhal-Burkard04, AGP05, AKP05;



i.e.,  $\varepsilon \approx \sqrt{A} (t_0 / \tau_c)^{1/2}$  (new) vs.  $\varepsilon \approx \sqrt{A} (t_0 / \tau_c)$  (old).

Even this weaker dependence on the ratio of the working period of a gate to the cutoff time scale may be spurious, since we expect the high frequency fluctuations to average out. However, it is hard to do better without using specific properties of the system Hamiltonian (e.g., the qubit frequencies).

One special Gaussian model where the sensitivity to high frequency noise is mild is pure dephasing noise, where all quantum gates are diagonal in the computational basis (as in AP07). For the dephasing/diagonal case, the faults commute with the system-bath evolution operator and can be propagated forward to the measurements. The diagrams can be summed explicitly, and only logarithmic dependence on the cutoff is found.

# Toward “realistic noise”

Noise models assumed by theorists are often highly idealized, at best crude approximations to the noise in actual devices. In formulating these models, one desires on the one hand to capture the essential features of realistic noise, but on the other hand to allow a succinct and elegant analysis of a computer’s reliability.

Seeking a balance between these two desiderata, we have proved a threshold theorem that applies to Gaussian quantum noise, a noise model which is physically well motivated and analytically tractable. Our result shows that (for this model) quantum computing is scalable if the noise correlation function obeys a suitable condition.

Compared to a previously derived threshold condition in terms of the norm of the system-bath Hamiltonian, this new condition has two advantages: it is expressed in terms of experimentally observable features of the noise, and it is less sensitive to high-frequency noise.

Though reduced, an annoying sensitivity to high-frequency noise still persists. Is this a mathematical technicality, or an actual potential obstacle to large scale fault-tolerant computing?

# Some further issues

- 1) Can we rigorously justify the “error phase randomization hypothesis” (error probability linear in number of gates)?
- 2) In Hamiltonian noise models, can we further reduce the sensitivity of threshold estimates to high-frequency noise, include non-Gaussian correlations, etc.?
- 3) What other schemes are scalable (besides concatenated codes and topological codes)? (For example, fault-tolerant adiabatic QC?)
- 4) Adapting fault-tolerance to properties of noise and specific algorithms.
- 5) Optimizing overhead cost.
- 6) Incorporating spin echo, dynamical decoupling, etc. into analysis of fault-tolerant protocols.
- 7) Realizing topological protection and topological processing
- 8) Self-correcting systems and devices (“finite-temperature topological order” in fewer than four spatial dimensions).
- 9) Systems engineering challenges (wires, power, cooling, ..)

# Quantum fault tolerance

Operating a large-scale quantum computer will be a grand scientific and engineering achievement.

Judicious application of the principles of fault-tolerant quantum computing will be the key to making it happen.

Fascinating connections with statistical physics, quantum many-body theory, device physics, and decoherence make the study of quantum fault tolerance highly rewarding.

We've got a long way to go.