Planar AdS/CFT: wrapping it up

Sakura Schäfer-Nameki
(Caltech)

IPMU Tokyo, 19. June 2008

N. Gromov, SSN, P. Vieiera, arxiv: 0801.3671, 0806/7.nnnn [hep-th]
Motivations

Goal: Establishing AdS/CFT quantitative tests, not relying on non-renormalization theorems

Obstructions: Recap:

\[ d = 4, \mathcal{N} = 4, SU(N_c) \text{ SYM} \]

\[ g_Y M, \quad \lambda = g_Y^2 N_c \]

Scaling dimensions \( \Delta \) of GIOs

Planar limit

\begin{align*}
\lambda & \ll 1 \\
\frac{1}{\sqrt{\lambda}} & - \text{corrections} \\
\lambda & = \infty
\end{align*}

Non-interacting strings

\[ \lambda \gg 1 \]

\[ \alpha' \text{-corrections} \]

\[ \alpha' = 0 \text{ (classical limit)} \]
Reasonable goal: Solving planar AdS/CFT (spectral AdS/CFT)

Immense progress: AdS/CFT as a one-parameter family of integrable models

\( \mathcal{N} = 4 \) dilatation operator

\( AdS_5 \times S^5 \) string energies

\[
\text{Diagonalize } \mathcal{D} \text{ acting on } \quad \text{Supercoset } \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}
\]

\[
\text{Tr}(ZWWZDZ \cdots) \quad \text{sigma-model}
\]

\[
\downarrow \quad \downarrow
\]

Integrability

\( \text{Diagonalize integrable } \quad \text{2d integrable QFT on cylinder} \)

\( \text{Spin-chain Hamiltonian} \quad \text{All-loop Bethe Ansatz and S-matrix } S = S(\lambda) \)

[Beisert, Hernández, López] [Beisert, Eden, Staudacher]

\[
\Rightarrow \Delta(\lambda) = E(\lambda)
\]
Large charge states:

- In agreement with 4-loop anomalous dims
- In agreement with $\alpha'$ corrections to spinning strings
- Interpolates correctly from $\lambda = 0$ to $\lambda = \infty$

Shortcomings:

- Integrability? S-matrix program assumes factorized scattering
  $\mathcal{N} = 4$: 1-loop [BS], 2-loop [Zwiebel]
  $AdS_5 \times S^5$: classical [Bena, Polchinski, Roiban], 1-loop [Berkovits], [Mikhailov, SSN]
- S-matrix describes asymptotic spectrum
  $\mathcal{N} = 4$ SYM: wrapping effects, breakdown for length $L \leq |\text{loops}|$
  4-loop Konishi: [KLRSV], [Fiamberti, Santambrogio, Sieg, Zanon], [Keeler, Mann]
  $AdS_5 \times S^5$: mismatch with $\alpha'$-corrections to string energies
  $\alpha'$-corrections: [SSN], [SSN, Zamaklar, Zarembo], [Janik, Lukowski], [Gromov, SSN, Vieiera]

Need to find systematic framework to include these finite-volume effects!
Plan

1. Introduction

2. $\mathcal{N} = 4$ SYM and integrability
   - How to use spin-chains to compute anomalous dimensions
   - All-loop spin-chain and S-matrix
   - Wrapping effects

3. $AdS_5 \times S^5$ and finite-size effect
   - S-matrix
   - Finite-size effects and Lüscher formulas

4. Efficient precision quantization in $AdS_5 \times S^5$
   - Classical curve and precision quantization
   - Quantization

5. Conclusions and Outlook
2. $\mathcal{N} = 4$ SYM and integrability

Dilatation operator $\mathcal{D}$, with eigenvalues $\Delta$

$$O(x)O(y) \sim \frac{C}{|x - y|^{2\Delta}}$$

Formally diverges. Wave-function renormalization $O^a_{\text{ren}} = Z^a_{\text{bare}} O^b_{\text{bare}}$

Determines anomalous dimension matrix by $\frac{dZ}{d\log \Lambda} Z^{-1}$. Eigenvectors are linear combinations that are multiplicatively renormalizable.

Consider $\mathfrak{su}(2)$ sector: $Z = \Phi_1 + i\Phi_2$ and $W = \Phi_3 + i\Phi_4$

$$O = \text{Tr}(ZZWZ \cdots WZWWZ \cdots W) + \text{permutations}$$

How to solve a hard problem? Map it to a known, solved problem!

Key insight of [Minahan, Zarembo]:

$$\text{Tr}(ZWZWWZWWZWZWWZW) \rightarrow \text{spin-chain}:
Then the dilatation operator at one-loop acts precisely like the Heisenberg XXX-Hamiltonian:

\[
\text{Tr}(ZZW \cdots W) \quad \rightarrow \quad |↑↑↓ \cdots ↓\rangle
\]

\[
\mathcal{D} \quad \rightarrow \quad H_{XXX} = \sum_{i=1}^{L} (1 - P_{i,i+1})
\]

"Nearest neighbour interaction"

\[
P_{i,i+1}(V_i \otimes V_{i+1}) = V_{i+1} \otimes V_i
\]

Groundstate: ferrogmanetic \(Tr Z^L\).

Diagonalization of Heisenberg Hamiltonian is a well-studied problem: S-matrix and Bethe ansatz.
2 → 2 scattering

\( W = \) excitation (magnon, \( \downarrow \)) with momentum \( p \) on the ground state 
\( \text{Tr} Z^L = | \uparrow \ldots \uparrow \rangle \).

Position space wave function

\[
|\Psi\rangle = \sum_{1 \leq x < y \leq L} \psi(x, y) |ZZWZ\cdots ZWZ\cdots Z\rangle
\]

Ansatz:

\[
\psi(x, y) = e^{ipx + iqy} + S(p, q)e^{ipy + qx}
\]

Schrödinger equation for \( H_{XXX} \) yields:

\[
E_{XXX} = 4 \sin^2(p/2) + 4 \sin^2(q/2) \quad \text{and} \quad S(p, q) = -\frac{1 + e^{i(p+q)} - 2e^{ip}}{1 + e^{i(p+q)} - 2e^{iq}}
\]
Factorized scattering

• 2 → 2 scattering:

\[ p + q = p' + q' \]
\[ E(p) + E(q) = E(p') + E(q') \]
\[ \Rightarrow (p', q') = (p, q) \text{ or } (q, p) \]

• \( n \rightarrow n \) scattering:

If there are \( n \) integrals of motion \( I_i \) then the same argument gives

\[ \sum_k p_k = \sum_k p'_k \]
\[ \sum_k I_i(p_k) = \sum_k I_i(p'_k) \]
\[ \Rightarrow p'_k = p_{\sigma(k)} \text{ for } \sigma \in S_n \]

In fact, we then have factorized scattering \( \sigma = \Pi_j \tau_j \).

In particular in models with infinite number of conserved charges the scattering is determined once \( E(p) \) and \( S(p, q) \) is known!
Periodicity and Bethe ansatz

\[ \psi(x, y) = \psi(y, x + L) \Rightarrow \]

\[ e^{ipL} = S(p, q) \quad e^{iqL} = S(p, q) \]

For general number of \( M \) excitations: Bethe equations

\[ e^{ip_i L} = \prod_{j \neq i}^M S(p_i, p_j) \]

Setting \( u_k = 1/2 \cot(p_k/2) \) (Bethe roots) they take the usual form

\[ \left( \frac{u_k + i/2}{u_k - i/2} \right)^L = \prod_{i=1}^M \frac{u_k - u_j + i}{u_k - u_j - i} \]

and the energy is

\[ E = \sum_{k=1}^M \frac{1}{u_k^2 + 1/4} \]
All-loop S-matrix

Amazingly this can be generalized to all operators of $\mathcal{N} = 4$ and asymptotically to all loops.

Loop-order: $L$ loops gives $L^{\text{th}}$ neighbour interacting spin-chain

Generic states: $\text{Tr}(ZWZYZWWWDZZ \cdots)$

Constituents: $\mathfrak{psu}(2, 2|4)$ field strength multiplet: $D^k \Phi_i, D^k \Psi, D^k F$

S-matrix picture: fix vacuum $\text{Tr} Z^L$ ($L$ large), other fields $\equiv$ excitations

Residual symmetry: $(\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)) \rtimes \mathbb{R}$

$S_{\mathfrak{su}(2|2)}$-matrix: $S^f_{i_1 i_2}$, where $i_1, i_2, f_1, f_2$ label each a $(2|2)$ representation.
\( su(2|2) \oplus su(2|2) \) fixes the S-matrix up to a scalar dressing factor \( \sigma \) \[\text{Beisert}\]

\[
S(p_k, p_j) = S_{su(2|2)}(p_k, p_j) S_{su(2|2)}(p_k, p_j) \sigma(p_k, p_j)
\]

Central charge determines dispersion relation

\[
\Delta - J = \sum_k \epsilon(p_k) = \sum_k \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}}
\]

Dressing factor: is fixed by crossing symmetry \[\text{Janik}\] and "experiments". Finally to determine energy of states we need a quantization condition on the momenta \( p_k \), aka Bethe ansatz equations

\[
e^{ip_i L} = \prod_{k \neq i} S(p_k, p_i)
\]

Plan of action:
1. Determine solutions \( p_k \) to the Bethe equations
2. Plug into \( \sum_k \epsilon(p_k) \)
3. Finished
The S-matrix approach is only valid \textit{asymptotically}, i.e. when free states can be prepared. \textbf{Wrapping interactions} violate this

\[
\text{spin-chain length } L \leq |\text{Loops}|
\]

Exemplified by length 4 Konishi-operator $\text{Tr}(\uparrow\downarrow\uparrow\downarrow - \uparrow\uparrow\downarrow\downarrow)$ at 4-loops:
up to 3-loops $\text{BAE} = \text{Feynman computation} \ [\text{KLOV}]$

\[
\Delta = 4 + 12g^2 - 48g^4 + 336g^6 + \Delta^{(4)}g^8 + \cdots
\]

$g^2 = \lambda/16\pi^2$, but at 4-loops:

\[
\Delta_{\text{BAE}}^{(4)} \neq \Delta^{(4)}_{\text{[KLRSV]}} \neq \Delta^{(4)}_{\text{[Fiamberti, Santambrogio, Sieg, Zanon]}} \neq \Delta^{(4)}_{\text{[Keeler, Mann]}}
\]

So far inconclusive, but appearance of $\zeta(5)$ in all explicit computations hint at invalidity of Bethe ansatz.

We will now see a similar phenomenon in the $\text{AdS}_5 \times S^5$ string, and propose some systematic framework to study these effects.
3. \( AdS_5 \times S^5 \) and finite-size effects

Take all-loop S-matrix seriously and assume it describes strong coupling, i.e. string dynamics as well.

- Light-cone Metsaev-Tseytlin action has \( su(2|2) \oplus su(2|2) \)
  \( \Rightarrow \) S-matrix equally fixed by Beisert’s analysis.

- All-loop dressing factor \( \sigma \) engineered such that
  \[ E_{\text{string}} = \sqrt{\lambda} E_0 + E_1 + O(1/\sqrt{\lambda}) \]
  agrees in infinite volume

However: similar mismatch between string energies and BAE prediction.

Length of the string

Consider large \( J = su(2) \) spin solutions. Uniform light-cone gauge:

\[ x^+ = \tau, \quad p_+ = 1 \Rightarrow P^+ = \frac{\sqrt{\lambda}}{2\pi} \int_0^L d\sigma = \text{Length}. \]

Light-cone coordinate \( p_+ = p_\phi \)

\[ \Rightarrow \quad \text{Length} = \frac{P^+}{\sqrt{\lambda}} = \frac{J}{\sqrt{\lambda}} \equiv g \]

At each order in \( 1/\sqrt{\lambda} \):

\[ E_i = E_i(g) \]
Finite-size corrections 1: Spinning String energies

Strings spinning on $\mathbb{R} \times S^3$  
Strings spinning on $AdS_3 \times S^1$

String energies: $E_0(j)$ agrees with BAE. One loop-shift is determined by sum over fluctuation frequencies [Frolov, Tseytlin, Tirziu,...]

\[ E_1(j) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{I} (-1)^{F_I} \Omega^{I}_n(j) \]

Expansion in $j$ is

\[ E_1(j) = \sum_{n} a_n \frac{1}{j^n} + \sum_{n} b_n e^{-2\pi j n} \]

and exponential terms are absent in BAE [SSN],[SSN, Zamaklar, Zarembo].
Finite-size corrections 2: Giant Magnons

GM = analog of single excitation on the spin-chain.

Classical solution of $\mathbb{R} \times S^2$ sigma-model [Hofman, Maldacena]

- $\Delta - J = \frac{\sqrt{\lambda}}{2\pi} \int_0^L d\sigma \mathcal{H}$
- $p = -\int_0^L d\sigma p_i x^i = \text{charge associated to translational invariance in } \sigma$

Determine $\Delta - J = \epsilon(p)$ yields dispersion relation

[Hofman, Maldacena], [Arutyunov, Frolov, Zamaklar], [Hatsuda, Suzuki], [Minahan, Sax]

\[
L = \infty : \quad \Delta - J = \epsilon_\infty(p) = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p}{2} \right| = \epsilon(p)_{O(\sqrt{\lambda})}
\]

\[
L < \infty : \quad \Delta - J = \epsilon_L(p) = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p}{2} \right| \left( 1 - \frac{4}{e^2} \left| \sin^3 \frac{p}{2} \right| e^{-2\pi \sin \frac{p}{2}} + \cdots \right)
\]

This is a classical effect, i.e. corrects $E_0$ by finite-volume terms.
General structure is

\[ \epsilon(p) - \epsilon_\infty(p) = \sqrt{\lambda} \delta \epsilon_{\text{classical}}(p) + \delta \epsilon_{1-\text{loop}}(p) + \frac{1}{\sqrt{\lambda}} \delta \epsilon_{2-\text{loop}} + O(1/\lambda). \]

So far:

\[ \sqrt{\lambda} \delta \epsilon_{\text{classical}}(p) = \sqrt{\frac{\lambda}{\pi}} \left| \sin \frac{p}{2} \right| \left( 1 - \frac{4}{e^2} \left| \sin \frac{p}{2} \right| e^{-2\pi \frac{g}{\sin p/2}} \right) \]

Computing the one-loop \( \alpha' = 1/\sqrt{\lambda} \) correction we find corrections of the order \( e^{-2\pi g} \) [Gromov, SSN, Vieira]

\[ \delta \epsilon_{1-\text{loop}}(p) = a_{1,0} e^{-2\pi g} + \sum_{n,m} a_{n,m}(\Delta) \exp (-n2\pi g) \exp \left( -m2\pi \frac{g}{\sin p/2} \right) \]

What is the physical interpretation of these corrections?
Lüscher formulas [Lüscher], [Klassen, Melzer]

Field-theoretic approach to compute leading finite-volume effects. Idea: 
**Infinite volume** 2-dim field theory, Euclidean Green’s function for 
elementary excitation/magnon $G_a(p)$ is

$$G_a(p) = \frac{1}{\epsilon_E^2 + \epsilon(p)^2 - \Sigma_a(p)}$$

$\epsilon_E$ = Euclidean energy, $\Sigma_a(p)$ = self-energy. Fix Res $\epsilon_E^2 G(p) = 1$
On-shell: $\epsilon_E^2 = \epsilon(p)^2$ and $\Sigma = \Sigma' = 0$.

In **finite volume** $L$, the self-energy $\Sigma_L(p)$ gets corrected by:
On the cylinder: average position space Green's function over \( \sigma \rightarrow \sigma + nL \). Momentum space, leading correction will be \( e^{ipL} \).

Leading order correction arises from keep only \( n = 1 \). E.g.

\[
= \sum_b \int_{\mathbb{R}} \frac{d^2 q}{(2\pi)^2} e^{iq^1L} G_{ab}(p) \Gamma^{aabb}(p, -p, q, -q)
\]

Move contour so that exponential suppresses integral. Picks up pole \( q^1 = q^* \) of \( G \), i.e. puts lines on-shell!

\[
= \sum_b \int \frac{dq^0}{2\pi} \frac{i}{\epsilon^2(q^*)} e^{-\epsilon q^* L} \Gamma^{aabb} = \sum_b \Gamma_{abab}
\]
Including all channels for particle with flavour $a$

\[= \int \frac{dq^0}{2\pi} \frac{i}{\epsilon_2(q^*)^2} e^{-|q^*|L} \sum_b G_{abab}(p, -p, q, -q)\]

$G_{abab}$ is the amputated, connected 4-point function.

Finite-size correction to dispersion relation $\delta\epsilon_L$ follows by

\[\epsilon_E^2 + \epsilon(p)^2 - \Sigma_L(p) = 0, \text{ where now on-shell } \epsilon_E^2 + (\epsilon(p) + \delta\epsilon_p)^2 = 0. \text{ Thus} \]

\[\delta\epsilon_L = -\frac{1}{2\epsilon(p)} \Sigma_L(p).\]

In integrable theories, this is related to the $S$-matrix. This yields the Lüscher F-term

\[\delta\epsilon^F(p) = -\frac{1}{2\epsilon(p)} \Sigma_L(p) = -\int_{\mathbb{R}} \frac{dq^0}{2\pi} \text{ (kin. factors) } e^{-iq^*L} \sum_b (-1)^{F_b} \left( S_{ba}(q^*, p) - 1 \right)\]
Extra contribution: first integral has both $G_b(q)$ and $G_c(q+p)$, thus neglected one term

$$\delta \epsilon^\mu(p) = -i \text{ (kin. factors)} e^{-i\tilde{q}^* L} \text{Res}_{q=\tilde{q}} \sum_b (-1)^F_b \left( S_{ba}^{ba}(q^*, p) - 1 \right)$$

$F$-term is virtual particle correcting $\Sigma$. $\mu$-term from bound state poles of the $S$-matrix.

For general dispersion relations derived by [Janik, Lukowski].
We can evaluate the Lüscher terms and compare them with the semi-classical string computation.

- Lüscher (µ-term) reproduces $\delta\epsilon_{\text{classical}}$ \cite{Janik, Lukowski}
  $\equiv$ contributions from bound state poles

- Lüscher (F-term) reproduces $a_{1,0}e^{-2\pi j}$ correction at one-loop
  \cite{Gromov, SSN, Vieira}
  $\equiv$ corrections due to virtual particles

- Subleading: general $a_{n,m}$ term in

  \[\delta\epsilon_{\text{1-loop}}(p) = a_{1,0}e^{-2\pi j} + \sum_{n,m} a_{n,m}(\Delta) \exp(-n2\pi j) \exp\left(-m2\pi \frac{g}{\sin p/2}\right)\]

  $\equiv$ $n$ virtual particle loops and $m$ splits into on-shell particles.

We will show next how to actually derive a large portion of these terms exactly from the string sigma-model, using the integrable structure.
Summary so far

1. S-matrix and Bethe ansatz seems to be correct to all orders in $\lambda$, as long as spin-chain/string are long enough

2. $\mathcal{N} = 4$: wrapping interactions spoil validity of Bethe ansatz if loop order is larger or equal to the length of the operator

3. $AdS_5 \times S^5$: exponentially suppressed terms (in $j$, i.e. length of the string) appear in the dispersion relation of the Giant Magnon and spinning string energies – classically and at one-loop

4. Systematic treatment of leading exponential terms by Lüscher formulas
4. Efficient Precision quantization in $AdS_5 \times S^5$

Semi-classical quantization can be performed directly in the sigma-model [Frolov, Tseytlin, Tirziu...], and from this approach one can derive the exponential terms [SSN] ⇒ very tedious.

Find efficient way to compute finite-volume terms exactly! Will teach us how to generalize/modify/extend Bethe ansatz equations.

Explicitly use classical/semi-classical integrability of the string sigma-model:

\[\infty \# \text{ conserved charges of the } AdS_5 \times S^5 \text{ string} \]
⇒ classical integrability [Bena, Polchinski, Roiban]

Aim: Determine classical solutions and their energies, and then semi-classically quantize as generically as possible.
Algebraic Curve

Classical integrability is ensured, if EOM can be written as the zero-curvature equation of a connection $J(x)$

$$dJ(x) - J(x) \wedge J(x) = 0$$

$x \in \mathbb{C}$ spectral parameter. Monodromy matrix

$$\Omega(x) = P \exp \left( \int A(x) \right)$$

Conserved charges follow by expanding $\text{Tr}\Omega(x)$. Eigenvalues: $e^{ip_i(x)}$ parametrize an algebraic curve. Essentially, because they satisfy the characteristic polynomial equation.

Point: classical solutions $\leftrightarrow$ algebraic curves
Principal chiral model

\[ g : \Sigma \to G = SU(2), SL(2) \]

[Kazakov, Marshakov, Minahan, Zarembo]

**Currents:** \[ j_\pm = g^{-1} \partial_\pm g \]

**E.O.M.:** \[ \partial_+ j_- + \partial_- j_+ = 0 \]
[\[ \partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0 \]

**Virasoro constraint:** \[ -\frac{1}{2} \text{Tr} j_\pm^2 = \kappa^2 \]

**Linear system/Lax pair:** \[ \mathcal{L} = \partial_\sigma + \frac{1}{2} \left( \frac{j_+}{1-x} - \frac{j_-}{1+x} \right) \]
[\[ \mathcal{M} = \partial_\tau + \frac{1}{2} \left( \frac{j_+}{1-x} + \frac{j_-}{1+x} \right) \]

**E.O.M. \iff** \[ \partial_\sigma \mathcal{M}_\tau - \partial_\tau \mathcal{L}_\sigma + [\mathcal{M}_\tau, \mathcal{L}_\sigma] = 0 \]
Algebraic curve

Monodromy matrix

\[ \Omega(x) = P \exp \left( - \int_0^{2\pi} d\sigma L_\sigma \right) = \begin{pmatrix} e^{ip(x)} & 0 \\ 0 & e^{-ip(x)} \end{pmatrix} \in SU(2) \]

\( \text{Tr} \Omega = 2 \cos(p(x)) \) independent of contour \( \Rightarrow p(x) \) is conserved.

Quasi-momenta \( p(x) \) are holomorphic in \( x \) except for poles at \( x = \pm 1 \) and branch-cuts \( C_k \).

- \( \text{Det} \Omega(x) = 1 \iff e^{ip(x^+)}e^{ip(x^-)} = 1 \) for \( x \in C_k \)
  i.e. for \( n_k \in \mathbb{Z} \)

\[ p(x) = p(x^+) + p(x^-) = 2\pi n_k, \quad z \in C_k. \]

- Filling fraction= A-cycle integral
  \[ S_{ij} = \oint_{C_{ij}} \left( 1 - \frac{1}{x^2} \right) p_i(x) \]
$AdS_5 \times S^5$ superstring algebraic curve

$\mathfrak{psu}(2,2|4)$ "mulatta" Dynkin diagram gives rise to $4|4$ 7-sheeted Riemann surface: $\otimes =$ fermionic roots.

$S^5$: $(\hat{1}, \tilde{3}), (\hat{1}, \tilde{4}), (\tilde{2}, \tilde{3}), (\tilde{2}, \tilde{4})$

$AdS_5$: $(\hat{1}, \tilde{3}), (\tilde{1}, \hat{4}), (\tilde{2}, \tilde{3}), (\tilde{2}, \tilde{4})$

Fermions: $(\tilde{1}, \hat{3}), (\hat{1}, \tilde{4}), (\tilde{2}, \hat{3}), (\hat{2}, \tilde{4})$

$(\hat{1}, \tilde{3}), (\hat{1}, \tilde{4}), (\tilde{2}, \tilde{3}), (\tilde{2}, \tilde{4})$
Quasimomenta:

\{ \tilde{p}_1(x) \mid \hat{p}_1(x), \hat{p}_2(x), \tilde{p}_2(x), \hat{p}_3(x), \tilde{p}_3(x), \hat{p}_4(x), \tilde{p}_4(x) \}

At the cuts/fermionic poles connecting sheets \((ij)\) "classical BAE"

\[ \phi_i(x) - \phi_j(x) = 2\pi n_{ij} \]

Asymptotics for \(x \to \infty\): connection becomes Noether currents, so \(p_i\) are related to global \(\mathfrak{psu}(2,2|4)\) charges, in particular the classical energy \(E\):

\[
\begin{pmatrix}
\hat{p}_1 \\
\hat{p}_2 \\
\hat{p}_3 \\
\hat{p}_4 \\
\tilde{p}_1 \\
\tilde{p}_2 \\
\tilde{p}_3 \\
\tilde{p}_4
\end{pmatrix}
\sim \frac{2\pi}{x\sqrt{\lambda}}
\begin{pmatrix}
+E - S_1 + S_2 \\
+E + S_1 - S_2 \\
-E - S_1 - S_2 \\
-E + S_1 + S_2 \\
+J_1 + J_2 - J_3 \\
+J_1 - J_2 + J_3 \\
-J_1 + J_2 + J_3 \\
-J_1 - J_2 - J_3
\end{pmatrix}
\]

Other constraints:

- Poles at \(x = \pm 1\): synchronized by Virasoro constraint.

- \(x \to 1/x\) acts as automorphism of \(\mathfrak{psu}(2,2|4)\):

\[ p_{1,2,3,4}(1/x) \to -p_{2,1,4,3}(x) \]
Quantizing the algebraic curve

Classical curve. Add small fluctuations, i.e. poles. Determine where they are localized and what the backreaction is,

\[ p_i(x) \to p_i(x) + \delta_n^{(ij)} p_i(x) \]

A single excitation of flavour \((ij)\) and mode number \(n\) is defined by shifting the filling fractions \(S_{ij} \to S_{ij} + 1\). This fixes [Beisert, Freyhult], [Gromov, Vieira]

\[ \delta_n^{(ij)} p_i(x) \sim \frac{\alpha(x_n^{ij})}{x - x_n^{ij}}, \quad \alpha(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1} \]

The position of the fluctuation pole \(x_n^{ij}\) is determined by classical BAE

\[ p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n_{ij} \]
Asymptotics at infinite are

$$\begin{pmatrix}
\delta \hat{p}_1 \\
\delta \hat{p}_2 \\
\delta \hat{p}_3 \\
\delta \hat{p}_4 \\
\end{pmatrix}
\sim \frac{4\pi}{x\sqrt{\lambda}}
\begin{pmatrix}
+\delta \Delta /2 & +N_{14} + N_{1\bar{3}} & +N_{\bar{1}3} + N_{14} \\
+\delta \Delta /2 & +N_{2\bar{3}} + N_{24} & +N_{24} + N_{23} \\
-\delta \Delta /2 & -N_{2\bar{3}} - N_{1\bar{3}} & -N_{1\bar{3}} - N_{2\bar{3}} \\
-\delta \Delta /2 & -N_{14} - N_{24} & -N_{24} - N_{14} \\
\end{pmatrix}
$$

- Synchronized poles at $x = \pm 1$
- $x \to 1/x$
- close to cuts: $\delta p \sim \partial_x p$

General strategy to determine one-loop energy shift:

1. Make ansatz for $\delta p_i$ that satisfies correct asymptotics and poles at $x_{n}^{ij}$, and $x = \pm 1$

2. Solve linear equations for undetermined constants and $\delta \Delta$
Efficient quantization of the algebraic curve

TBF\[Gromov, SSN, Vieira\]

The above procedure is already more efficient than semi-classical sigma-model approach à la Frolov-Tseytlin.

Can do better! Why? For more complicated solutions this is crucial to obtain exact, finite-size spectrum. E.g. GM as 2-cut solution.

Use $x \rightarrow 1/x$ symmetry, then from one frequency in $S^3$ and one in $AdS_5$

$$\tilde{\Omega}_n = \Omega_n^{\hat{2}\hat{3}}, \quad \hat{\Omega}_n = \Omega_n^{\hat{2}\hat{3}}$$

we can determine all others as linear combinations.

NB: this is different from quantization in subsectors!

Let us consider an example.
Simple $S^3 \times \mathbb{R}$

Classical energy: $\kappa = \frac{E}{\sqrt{\lambda}} = \sqrt{j^2 + m^2}$. $J$ is spin, $m$ is winding on $S^3$. The classical solution is determined by

$$p_1 = p_2 = -p_3 = -p_4 = \pm \frac{2\pi x}{x^2 - 1} \kappa$$

$$p_1 = + \frac{2\pi x}{x^2 - 1} \sqrt{j^2 + \frac{m^2}{x^2}}$$

$$p_2 = + \frac{2\pi x}{x^2 - 1} \sqrt{j^2 + m^2 x^2} - 2\pi m$$

$$p_3 = - \frac{2\pi x}{x^2 - 1} \sqrt{j^2 + m^2 x^2} + 2\pi m$$

$$p_4 = - \frac{2\pi x}{x^2 - 1} \sqrt{j^2 + \frac{m^2}{x^2}}.$$

Check: Asymptotics, poles, $x \to 1/x$. 
Input fluctuations: $\Omega_n = \Omega(x_n)$

$$\hat{\Omega}^{\tilde{3}\tilde{3}}(x_n) = \frac{2m + n\tilde{2}\tilde{3}}{\kappa x_n} = \frac{2m + \frac{p_2 - p_3}{2\pi}}{\kappa x_n} = \frac{2\sqrt{m^2 x_n^2 + g^2}}{(x_n^2 - 1) \sqrt{m^2 + g^2}}$$

$$\hat{\Omega}^{\tilde{4}\tilde{4}}(x_n) = \frac{2}{x_n^2 - 1}$$

Tracing back the $x \to 1/x$ symmetry of the quasi-momenta and $\delta p$, we can show that off-shell frequencies $\Omega(x)$, where $\Omega(x)|_{x=x_n} = \Omega_n$ satisfy

$$\Omega^{\tilde{1}\tilde{4}}(x) = -\Omega^{\tilde{3}\tilde{3}}(1/x) + \text{const.} , \quad \Omega^{\tilde{4}\tilde{4}}(x) = -\Omega^{\tilde{3}\tilde{3}}(1/x) + \text{const.}$$

Remaining bosonic fluctuations are obtained by linear combinations of the off-shell frequencies

$$\Omega^{ij}(y) = \frac{1}{2} \left( \Omega^{ii'}(y) + \Omega^{jj'}(y) \right)$$

where

$$(\tilde{1}, \tilde{2}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{3}, \tilde{4})' = (\hat{1}, \hat{3}, \hat{4}, \hat{3}, \hat{2}, \hat{1}, \tilde{2}, \tilde{1}).$$
Applied to the $S^3 \times \mathbb{R}$ solution, we obtain all the well-known frequencies only in terms of the frequencies $\tilde{\Omega}$ and $\hat{\Omega}$

$$\Omega^{14}(y) = -\Omega^{23} (1/y) - 2 \frac{\partial E}{\partial j}$$

$$\Omega^{24}(y) = \Omega^{13}(y) = \frac{1}{2} \left( \Omega^{23}(y) + \Omega^{14}(y) \right) = \frac{1}{2} \left( \Omega^{23}(y) - \Omega^{23} (1/y) \right) - \frac{\partial E}{\partial j}$$

$$\Omega^{\hat{1}4}(y) = -\Omega^{\hat{2}3} (1/y) - 2$$

$$\Omega^{\hat{2}4}(y) = \Omega^{\hat{1}3}(y) = \frac{1}{2} \left( \Omega^{\hat{2}3}(y) + \Omega^{\hat{1}4}(y) \right) = \frac{1}{2} \left( \Omega^{\hat{2}3}(y) - \Omega^{\hat{2}3} (1/y) \right) - 1 - \frac{\partial E}{\partial j}$$

$$\Omega^{\hat{2}4}(y) = \Omega^{\hat{1}3}(y) = \frac{1}{2} \left( \Omega^{\hat{2}3}(y) + \Omega^{\hat{1}4}(y) \right) = \frac{1}{2} \left( \Omega^{\hat{2}3}(y) - \Omega^{\hat{2}3} (1/y) \right) - \frac{\partial E}{\partial j}$$

$$\Omega^{\hat{1}4}(y) = \Omega^{\hat{1}4}(y) = \frac{1}{2} \left( \Omega^{\hat{1}4}(y) + \Omega^{\hat{1}4}(y) \right) = \frac{1}{2} \left( -\Omega^{\hat{2}3} (1/y) - \Omega^{\hat{2}3} (1/y) \right) - 1 - \frac{\partial E}{\partial j}$$

$$\Omega^{\hat{2}3}(y) = \Omega^{\hat{2}3}(y) = \frac{1}{2} \left( \Omega^{\hat{2}3}(y) + \Omega^{\hat{2}3}(y) \right)$$
Summary: Efficient precision quantization

Finally the energy shift is computed by

- Solve for one $\tilde{\Omega}$ and one $\hat{\Omega}$
- Solve for the pole positions $x_{n}^{ij}$ (trivial)

\[ p_i(x_{n}^{ij}) - p_j(x_{n}^{ij}) = 2\pi n_{ij} \]

- Evaluating the plug-in formula above for the off-shell frequencies with arbitrary polarization $\Omega^{ij}(x)$ and $x = x_{n}^{ij}$ (trivial)

- $\delta\Delta = \sum_{n \in \mathbb{Z}} \sum_{ij} (-1)^{F_{ij}} \Omega_{n}^{ij}$ (finished)

Application: GM as a 2-cut solution. Subleading exponential corrections.

In this way: get complete control over exponential terms at one-loop

\[ \delta\Delta = \sum_{n \in \mathbb{Z}} \sum_{ij} (-1)^{F_{ij}} \Omega_{n}^{ij} = \int dncot(\pi n) \sum_{ij} (-1)^{F_{ij}} \Omega_{n}^{ij} \]

and expanding cotangent.
5. Wrapping up: Conclusions and Outlook

Summary:

- **Finite-size effects** are one crucial missing piece to be understood on the way to solve planar AdS/CFT
- 2-d field theory approach for the string: **Lüsch}er formulas** reproduce exponential terms in string energies
- Exponential terms at one-loop in $\alpha'$: efficient precision

Future directions:

- Application of **Lüscher formulas to Konishi**
- **Beyond leading exponential corrections:**
  From the algebraic curve: no problem at one-loop.
  Correct BAE to incorporate these
- **TBA?**
- Find more **formal evidence for integrability**
Thank you