Planar AdS/CFT: wrapping it up

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N. Gromov, SSN, P. Vieiera, `arxiv: 0801.3671, 0806/7.nnnn [hep-th]`

Motivations

Goal: Establishing AdS/CFT

quantitative tests, not relying on non-renormalization theorems

Obstructions: Recap:

$\boxed{d=4,\ \mathcal{N}=4,\ SU(N_c)\ \text{SYM}}$	$\boxed{\text{IIB on } AdS_5\times S^5}$
$g_{YM}\,,\qquad \lambda=g_{YM}^2N_c$	$g_s=g_{YM}^2\,,\qquad \alpha'=\frac{R^2}{\sqrt{\lambda}}$
Scaling dimensions Δ of GIOs	Energies E of string states
Planar limit	Non-interacting strings
$\lambda\ll 1$	$\lambda\gg 1$
$\frac{1}{\sqrt{\lambda}}$-corrections	α'-corrections
$\lambda=\infty$	$\alpha'=0$ (classical limit)

Reasonable goal: Solving planar AdS/CFT (spectral AdS/CFT)

Immense progress: AdS/CFT as a one-parameter family of integrable models

$\boxed{\mathcal{N} = 4 \text{ dilatation operator}}$

Diagonalize $\mathcal{D}$ acting on

$\mathrm{Tr}(ZWWZDZ\cdots)$

$\downarrow$

Diagonalize integrable

Spin-chain Hamiltonian

$\searrow$

$\boxed{\text{Integrability}}$

$\boxed{AdS_5 \times S^5 \text{ string energies}}$

Supercoset $\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$

sigma-model

$\downarrow$

2d integrable QFT on cylinder

$\swarrow$

All-loop Bethe Ansatz and S-matrix $S = S(\lambda)$

[Beisert, Hernández, López] [Beisert, Eden, Staudacher]

$$\Rightarrow \; \Delta(\lambda) = E(\lambda)$$

Large charge states:

- In agreement with 4-loop anomalous dims
- In agreement with α' corrections to spinning strings
- interpolates correctly from $\lambda = 0$ to $\lambda = \infty$!

Shortcomings:

- Integrability? S-matrix program *assumes* factorized scattering

  $\mathcal{N} = 4$: 1-loop [BS], 2-loop [Zwiebel]

  $AdS_5 \times S^5$: classical [Bena, Polchinski, Roiban], 1-loop [Berkovits], [Mikhailov, SSN]

- S-matrix describes asymptotic spectrum

  $\mathcal{N} = 4$ SYM: wrapping effects, breakdown for length $L \leq |loops|$

  4-loop Konishi: [KLRSV], [Fiamberti, Santambrogio, Sieg, Zanon], [Keeler, Mann]

  $AdS_5 \times S^5$: mismatch with α'-corrections to string energies

  α'-corrections: [SSN], [SSN, Zamaklar, Zarembo], [Janik, Lukowski], [Gromov, SSN, Vieiera]

Need to find systematic framework to include these finite-volume effects!

Plan

1. Introduction

2. $\mathcal{N} = 4$ SYM and integrability

   - How to use spin-chains to compute anomalous dimensions
   - All-loop spin-chain and S-matrix
   - Wrapping effects

3. $AdS_5 \times S^5$ and finite-size effect

   - S-matrix
   - Finite-size effects and Lüscher formulas

4. Efficient precision quantization in $AdS_5 \times S^5$

   - Classical curve and precision quantization
   - Quantization

5. Conclusions and Outlook

2. $\mathcal{N} = 4$ SYM and integrability

Dilatation operator $\mathcal{D}$, with eigenvalues Δ

$$\mathcal{O}(x)\mathcal{O}(y) \sim \frac{C}{|x-y|^{2\Delta}}$$

Formally diverges. Wave-function renormalization $\mathcal{O}^a_{ren} = Z^a_b \mathcal{O}^b_{bare}$
Determines anomalous dimension matrix by $\frac{dZ}{d\log\Lambda} Z^{-1}$. Eigenvectors are linear combinations that are multiplicatively renormalizable.

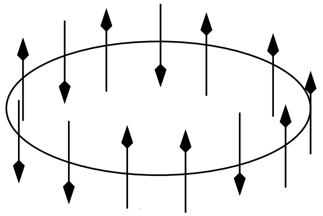
Consider $\mathfrak{su}(2)$ sector: $Z = \Phi_1 + i\Phi_2$ and $W = \Phi_3 + i\Phi_4$

$$\mathcal{O} = \mathrm{Tr}(ZZWZ\cdots WZWWZ\cdots W) + \text{permutations}$$

How to solve a hard problem? Map it to a known, solved problem!

Key insight of [Minahan, Zarembo]:

$\mathrm{Tr}(ZWZZZZWZWZWWZ) \rightarrow$ spin-chain:



Then the dilatation operator at one-loop acts precisely like the Heisenberg XXX-Hamiltonian:

$$\begin{aligned} \mathrm{Tr}(ZZW\cdots W) \quad &\longrightarrow \quad |\uparrow\uparrow\downarrow\cdots\downarrow\rangle \\ \mathcal{D} \quad &\longrightarrow \quad H_{XXX} = \sum_{i=1}^{L}(1 - P_{i,i+1}) \end{aligned}$$

"Nearest neighbour interaction"

$$P_{i,i+1}(V_i \otimes V_{i+1}) = V_{i+1} \otimes V_i$$

Groundstate: ferrogmanetic TrZ^L.

Diagonalization of Heisenberg Hamiltonian is a well-studied problem: S-matrix and Bethe ansatz.

$2 \longrightarrow 2$ scattering

$W =$ excitation (magnon, $\downarrow$) with momentum p on the ground state $\mathrm{Tr} Z^L = | \uparrow \cdots \uparrow \rangle$.

Position space wave function

$$|\Psi\rangle = \sum_{1 \le x < y \le L} \psi(x,y) | Z \cdots Z W Z \cdots Z W Z \cdots Z \rangle$$

Ansatz:

$$\psi(x,y) = e^{ipx+iqy} + S(p,q) e^{ipy+qx}$$

Schrödinger equation for H_{XXX} yields:

$$E_{XXX} = 4 \sin^2(p/2) + 4 \sin^2(q/2) \qquad \text{and} \qquad S(p,q) = -\frac{1 + e^{i(p+q)} - 2e^{ip}}{1 + e^{i(p+q)} - 2e^{iq}}$$

Factorized scattering

- $2 \to 2$ scattering:

$$\left.\begin{aligned} p+q &= p'+q' \\ E(p)+E(q) &= E(p')+E(q') \end{aligned}\right. \Rightarrow (p',q') = (p,q) \text{ or } (q,p)$$

- $n \to n$ scattering:

If there are n integrals of motion I_i then the same argument gives

$$\left.\begin{aligned} \Sigma_k\, p_k &= \Sigma_k\, p'_k \\ \Sigma_k\, I_i(p_k) &= \Sigma_k\, I_i(p'_k) \end{aligned}\right. \Rightarrow p'_k = p_{\sigma(k)} \text{ for } \sigma \in S_n$$

In fact, we then have factorized scattering $\sigma = \Pi_j \tau_j$.

In particular in models with infinite number of conserved charges the scattering is determined once $E(p)$ and $S(p,q)$ is known!

Periodicity and Bethe ansatz

$\psi(x, y) = \psi(y, x + L) \Rightarrow$

$$e^{ipL} = S(p, q) \qquad\qquad e^{iqL} = S(p, q)$$

For general number of M exitations: Bethe equations

$$e^{ip_i L} = \prod_{j \neq i}^{M} S(p_i, p_j)$$

Setting $u_k = 1/2 \cot(p_k/2)$ (Bethe roots) they take the usual form

$$\left(\frac{u_k + i/2}{u_k - i/2} \right)^L = \prod_{i=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}$$

and the energy is

$$E = \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4}$$

All-loop S-matrix

Amazingly this can be generalized to all operators of $\mathcal{N} = 4$ and asymptotically to all loops.

Loop-order: L loops gives L^{th} neighbour interacting spin-chain

Generic states: $\text{Tr}(ZWWZY\Psi ZZWWDZZ\cdots)$

Constituents: $\mathfrak{psu}(2,2|4)$ field strength multiplet: $\mathcal{D}^k\Phi_i, \mathcal{D}^k\Psi, \mathcal{D}^k\mathcal{F}$

S-matrix picture: fix vacuum $\text{Tr}Z^L$ (L large), other fields $\equiv$ excitations

Residual symmetry: $\big(\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)\big) \ltimes \mathbb{R}$

$S_{\mathfrak{su}(2|2)}$-matrix: $S^{f_1 f_2}_{i_1 i_2}$, where i_1, i_2, f_1, f_2 label each a $(2|2)$ representation.

$\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ fixes the S-matrix up to a scalar dressing factor σ [Beisert]

$$S(p_k, p_j) = S_{\mathfrak{su}(2|2)}(p_k, p_j) S_{\mathfrak{su}(2|2)}(p_k, p_j)\, \sigma(p_k, p_j)$$

Central charge determines dispersion relation

$$\Delta - J = \sum_k \epsilon(p_k) = \sum_k \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_k}{2}}$$

Dressing factor: is fixed by crossing symmetry [Janik] and ”experiments”. Finally to determine energy of states we need a quantization condition on the momenta p_k, aka Bethe ansatz equations

$$e^{ip_i L} = \prod_{k \neq i} S(p_k, p_i)$$

Plan of action:

1. Determine solutions p_k to the Bethe equations
2. Plug into $\sum_k \epsilon(p_k)$
3. Finished

Really?

The S-matrix approach is only valid asymptotically, i.e. when free states can be prepared.

Wrapping interactions violate this

$$\text{spin-chain length } L \leq |\text{Loops}|$$

Exemplified by length 4 Konishi-operator $\mathrm{Tr}(\uparrow\downarrow\uparrow\downarrow - \uparrow\uparrow\downarrow\downarrow)$ at 4-loops: up to 3-loops BAE = Feynman computation [KLOV]

$$\Delta = 4 + 12g^2 - 48g^4 + 336g^6 + \Delta^{(4)}g^8 + \cdots$$

$g^2 = \lambda/16\pi^2$, but at 4-loops:

$$\Delta^{(4)}_{\text{BAE}} \neq \Delta^{(4)}_{\text{[KLRSV]}} \neq \Delta^{(4)}_{\text{[Fiamberti, Santambrogio, Sieg, Zanon]}} \neq \Delta^{(4)}_{\text{[Keeler, Mann]}}$$

So far inconclusive, but appearance of $\zeta(5)$ in all explicit computations hint at invalidity of Bethe ansatz.

We will now see a similar phenomenon in the $AdS_5 \times S^5$ string, and propose some systematic framework to study these effects.

3. $AdS_5 \times S^5$ and finite-size effects

Take all-loop S-matrix seriously and assume it describes strong coupling, i.e. string dynamics as well.

- Light-cone Metsaev-Tseytlin action has $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ $\Rightarrow$ S-matrix equally fixed by Beisert's analysis.
- All-loop dressing factor σ engineered such that $E_{string} = \sqrt{\lambda}E_0 + E_1 + O(1/\sqrt{\lambda})$ agrees in infinite volume

However: similar mismatch between string energies and BAE prediction.

Length of the string

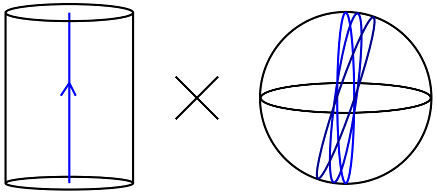
Consider large $J = \mathfrak{su}(2)$ spin solutions. Uniform light-cone gauge: $x^+ = \tau$, $p_+ = 1 \Rightarrow P^+ = \frac{\sqrt{\lambda}}{2\pi}\int_0^L d\sigma =$ Length. Light-cone coordinate $p_+ = p_\phi$

$$\Rightarrow \quad \text{Length} = \frac{P^+}{\sqrt{\lambda}} = \frac{J}{\sqrt{\lambda}} \equiv \mathcal{J}$$

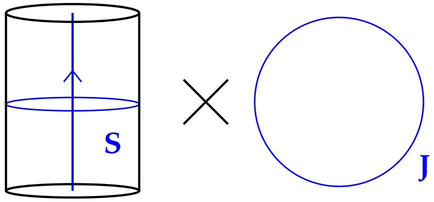
At each order in $1/\sqrt{\lambda}$: $\quad E_i = E_i(\mathcal{J})$

Finite-size corrections 1: Spinning String energies

Strings spinning on $\mathbb{R} \times S^3$



Strings spinning on $AdS_3 \times S^1$



String energies: $E_0(\mathcal{J})$ agrees with BAE. One loop-shift is determined by sum over fluctuation frequencies [Frolov, Tseytlin, Tirziu, ...]

$$E_1(\mathcal{J}) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_I (-1)^{F_I} \Omega_n^I(\mathcal{J})$$

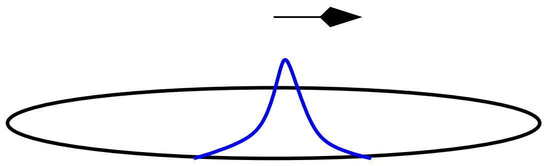
Expansion in $\mathcal{J}$ is

$$E_1(\mathcal{J}) = \sum_n a_n \frac{1}{\mathcal{J}^n} + \sum_n b_n e^{-2\pi \mathcal{J} n}$$

and exponential terms are absent in BAE [SSN],[SSN, Zamaklar, Zarembo].

Finite-size corrections 2: Giant Magnons

GM = analog of single excitation on the spin-chain.



Classical solution of $\mathbb{R} \times S^2$ sigma-model [Hofman, Maldacena]

- $\Delta - J = \frac{\sqrt{\lambda}}{2\pi} \int_0^L d\sigma \mathcal{H}$
- $p = -\int_0^L d\sigma p_i x^{i\prime}$ = charge associated to translational invariance in σ

Determine $\Delta - J = \epsilon(p)$ yields dispersion relation

[Hofman, Maldacena], [Arutyunov, Frolov, Zamaklar], [Hatsuda, Suzuki], [Minahan, Sax]

$$L = \infty : \quad \Delta - J = \epsilon_\infty(p) = \frac{\sqrt{\lambda}}{\pi} |\sin \frac{p}{2}| = \epsilon(p)|_{O\sqrt{(\lambda)}}$$

$$L < \infty : \quad \Delta - J = \epsilon_L(p) \ = \frac{\sqrt{\lambda}}{\pi} |\sin \frac{p}{2}| \left(1 - \frac{4}{e^2} |\sin^3 \frac{p}{2}| e^{-2\pi \frac{\mathcal{J}}{\sin p/2}} + \cdots \right)$$

This is a classical effect, i.e. corrects E_0 by finite-volume terms.

General structure is

$$\epsilon(p) - \epsilon_{\infty}(p) = \sqrt{\lambda}\,\delta\epsilon_{classical}(p) + \delta\epsilon_{1-loop}(p) + \frac{1}{\sqrt{\lambda}}\delta\epsilon_{2-loop} + O(1/\lambda)\,.$$

So far:

$$\sqrt{\lambda}\,\delta\epsilon_{classical}(p) = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right|\left(1 - \frac{4}{e^2}\left|\sin^3\frac{p}{2}\right| e^{-2\pi\frac{\mathcal{J}}{\sin p/2}}\right)$$

Computing the one-loop $\alpha' = 1/\sqrt{\lambda}$ correction we find corrections of the order $e^{-2\pi\mathcal{J}}$ [Gromov, SSN, Vieira]

$$\delta\epsilon_{1-loop}(p) = a_{1,0}e^{-2\pi\mathcal{J}} + \sum_{n,m} a_{n,m}(\Delta)\exp\left(-n2\pi\mathcal{J}\right)\exp\left(-m2\pi\frac{\mathcal{J}}{\sin p/2}\right)$$

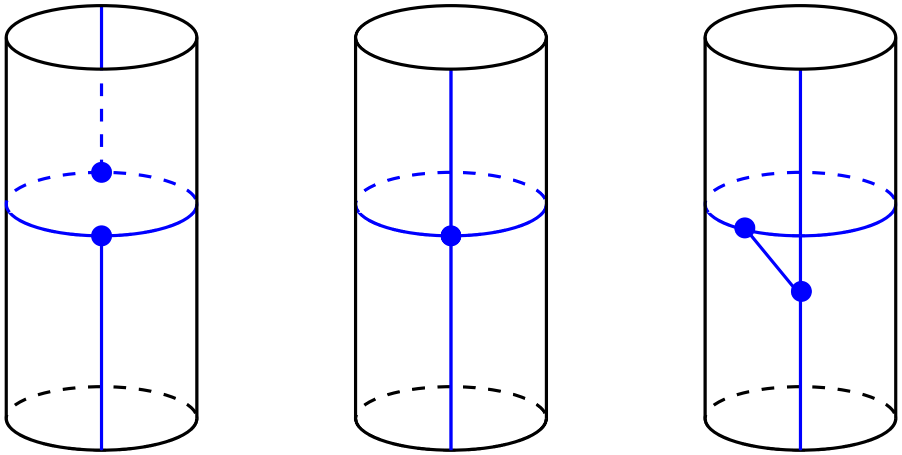
What is the physical interpretation of these corrections?

Lüscher formulas [Lüscher], [Klassen, Melzer]

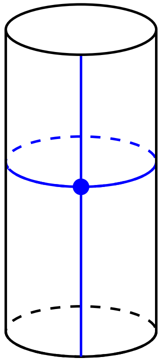
Field-theoretic approach to compute leading finite-volume effects. Idea: Infinite volume 2-dim field theory, Euclidean Green's function for elementary excitation/magnon $G_a(p)$ is

$$G_a(p) = \frac{1}{\epsilon_E^2 + \epsilon(p)^2 - \Sigma_a(p)}$$

ϵ_E= Euclidean energy, $\Sigma_a(p)$ = self-energy. Fix $\mathrm{Res}_{\epsilon_E^2} G(p) = 1$
On-shell: $\epsilon_E^2 = \epsilon(p)^2$ and $\Sigma = \Sigma' = 0$.
In finite volume L, the self-energy $\Sigma_L(p)$ gets corrected by:

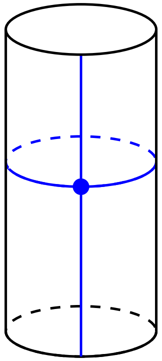


On the cylinder: average position space Green's function over $\sigma \to \sigma + nL$. Momentum space, leading correction will be e^{ipL} . Leading order correction arises from keep only $n = 1$. E.g.

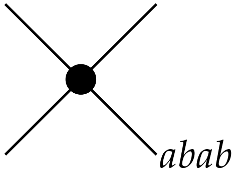


$$= \sum_b \int_{\mathbb{R}} \frac{d^2q}{(2\pi)^2} e^{iq^1 L} G_{ab}(p) \Gamma^{aabb}(p, -p, q, -q)$$

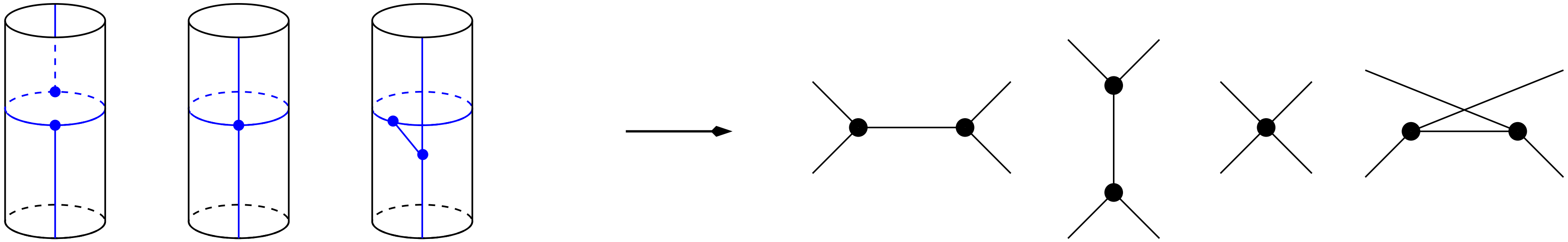
Move contour so that exponential suppresses integral. Picks up pole $q^1 = q^*$ of G, i.e. puts lines on-shell!



$$= \sum_b \int \frac{dq^0}{2\pi} \frac{i}{\epsilon^2 (q^*)'} e^{-|q^*|L} \Gamma^{aabb} = \sum_b$$



Including all channels for particle with flavour a



$$= \int \frac{dq^0}{2\pi} \frac{i}{\epsilon^2(q^*)'} e^{-|q^*|L} \sum_b G_{abab}(p, -p, q, -q)$$

G_{abab} is the amputated, connected 4-point function.

Finite-size correction to dispersion relation $\delta\epsilon_L$ follows by $\epsilon_E^2 + \epsilon(p)^2 - \Sigma_L(p) = 0$, where now on-shell $\epsilon_E^2 + (\epsilon(p) + \delta\epsilon_p)^2 = 0$. Thus

$$\delta\epsilon_L = -\frac{1}{2\epsilon(p)} \Sigma_L(p)\,.$$

In integrable theories, this is related to the S-matrix. This yields the Lüscher F-term

$$\delta\epsilon^F(p) = -\frac{1}{2\epsilon(p)} \Sigma_L(p) = -\int_{\mathbb{R}} \frac{dq^0}{2\pi} \,(\text{kin. factors})\, e^{-iq^*L} \sum_b (-1)^{F_b} \left(S_{ba}^{ba}(q^*, p) - 1 \right)$$

Extra contribution: first integral has both $G_b(q)$ and $G_c(q+p)$, thus neglected one term

$$\delta\epsilon^{\mu}(p) = -i\,(\text{kin. factors})\,e^{-i\tilde{q}^{*}L}\operatorname{Res}_{q=\tilde{q}}\sum_{b}(-1)^{F_b}\left(S_{ba}^{ba}(q^{*},p)-1\right)$$

F-term is virtual particle correcting Σ. μ-term from bound state poles of the S-matrix.

For general dispersion relations derived by [Janik, Lukowski].

We can evaluate the Lüscher terms and compare them with the semi-classical string computation.

- Lüscher (μ-term) reproduces $\delta\epsilon_{classical}$ [Janik, Lukowski]
  $\equiv$ contributions from bound state poles
- Lüscher (F-term) reproduces $a_{1,0}e^{-2\pi\mathcal{J}}$ correction at one-loop [Gromov, SSN, Vieira]
  $\equiv$corrections due to virtual particles
- Subleading: general $a_{n,m}$ term in

$$\delta\epsilon_{1-loop}(p) = a_{1,0}e^{-2\pi\mathcal{J}} + \sum_{n,m} a_{n,m}(\Delta)\exp\left(-n2\pi\mathcal{J}\right)\exp\left(-m2\pi\frac{\mathcal{J}}{\sin p/2}\right)$$

$\equiv$ n virtual particle loops and m splits into on-shell particles.

We will show next how to actually derive a large portion of these terms exactly from the string sigma-model, using the integrable structure.

Summary so far

1. S-matrix and Bethe ansatz seems to be correct to all orders in λ, as long as spin-chain/string are long enough
2. $\mathcal{N} = 4$: wrapping interactions spoil validity of Bethe ansatz if loop order is larger or equal to the length of the operator
3. $AdS_5 \times S^5$: exponentially suppressed terms (in $\mathcal{J}$, i.e. length of the string) appear in the dispersion relation of the Giant Magnon and spinning string energies – classically and at one-loop
4. Systematic treatment of leading exponential terms by Lüscher formulas

4. Efficient Precision quantization in $AdS_5 \times S^5$

Semi-classical quantization can be performed directly in the sigma-model [Frolov, Tseytlin, Tirziu...], and from this approach one can derive the exponential terms [SSN]
⇒ very tedious.

Find efficient way to compute finite-volume terms exactly! Will teach us how to generalize/modify/extend Bethe ansatz equations.

Explicitly use classical/semi-classical integrability of the string sigma-model:

∞# conserved charges of the $AdS_5 \times S^5$ string
⇒ classical integrability [Bena, Polchinski, Roiban]

Aim: Determine classical solutions and their energies, and then semi-classically quantize as generically as possible.

Algebraic Curve

Classical integrability is ensured, if EOM can be written as the zero-curvature equation of a connection $J(x)$

$$dJ(x) - J(x) \wedge J(x) = 0$$

$x \in \mathbb{C}$ spectral parameter. Monodromy matrix

$$\Omega(x) = P \exp\left(\int A(x) \right)$$

Conserved charges follow by expanding $\mathrm{Tr}\Omega(x)$.

Eigenvalues: $e^{ip_i(x)}$ parametrize an algebraic curve. Essentially, because they satisfy the characteristic polynomial equation.

Point: classical solutions $\overset{1:1}{\longleftrightarrow}$ algebraic curves

Principal chiral model

$$g : \Sigma \rightarrow G = SU(2),\ SL(2)$$

[Kazakov, Marshakov, Minahan, Zarembo]

Currents: $j_\pm = g^{-1}\partial_\pm g$

E.O.M: $\partial_+ j_- + \partial_- j_+ = 0$

$\partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0$

Virasoro constraint: $-\frac{1}{2}\mathrm{Tr}\, j_\pm^2 = \kappa^2$

Linear system/Lax pair: $\mathcal{L} = \partial_\sigma + \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right)$

$\mathcal{M} = \partial_\tau + \frac{1}{2}\left(\frac{j_+}{1-x} + \frac{j_-}{1+x}\right)$

E.O.M. ⇔ $\partial_\sigma \mathcal{M}_\tau - \partial_\tau \mathcal{L}_\sigma + [\mathcal{M}_\tau, \mathcal{L}_\sigma] = 0$

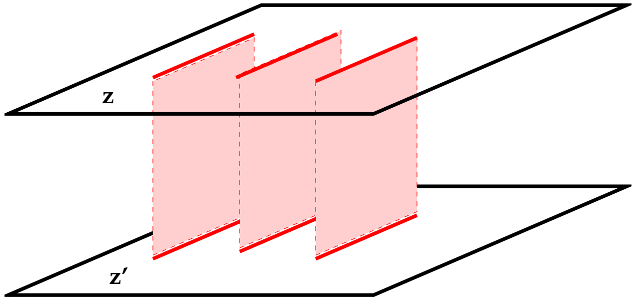
Algebraic curve

Monodromy matrix

$$\Omega(x) = P\exp\left(-\int_0^{2\pi} d\sigma\, \mathcal{L}_\sigma\right) \cong \begin{pmatrix} e^{ip(x)} & 0 \\ 0 & e^{-ip(x)} \end{pmatrix} \in SU(2)$$

$\mathrm{Tr}\Omega = 2\cos(p(x))$ independent of contour $\Rightarrow p(x)$ is conserved.

Quasi-momenta $p(x)$ are holomorphic in x except for poles at $x = \pm 1$ and branch-cuts $\mathcal{C}_k$.



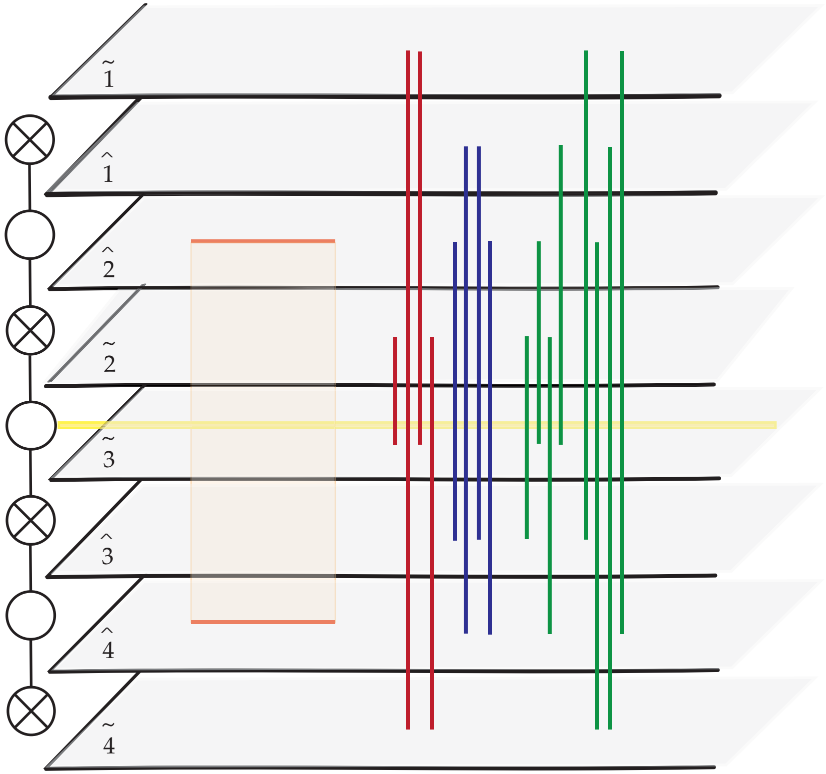
- $\mathrm{Det}\Omega(x) = 1 \Leftrightarrow e^{ip(x^+)}e^{ip(x^-)} = 1$ for $x \in \mathcal{C}_k$ *i.e.* for $n_k \in \mathbb{Z}$

$$\boxed{\not{p}(x) = p(x^+) + p(x^-) = 2\pi n_k\,, \qquad z \in \mathcal{C}_k\,.}$$

- Filling fraction= A-cycle integral $S_{ij} = \oint_{\mathcal{C}_{ij}} \left(1 - \frac{1}{x^2}\right) p_i(x)$

$AdS_5 \times S^5$ superstring algebraic curve

$\mathfrak{psu}(2,2|4)$ ”mulatta” Dynkin diagram gives rise to $\mathbf{4}|\mathbf{4}$ 7-sheeted Riemann surface: $\otimes$ = fermionic roots.



S^5 : $(\tilde{1},\tilde{3}),(\tilde{1},\tilde{4}),(\tilde{2},\tilde{3}),(\tilde{2},\tilde{4})$

AdS_5 : $(\hat{1},\hat{3}),(\hat{1},\hat{4}),(\hat{2},\hat{3}),(\hat{2},\hat{4})$

Fermions : $(\tilde{1},\hat{3}),(\tilde{1},\hat{4}),(\tilde{2},\hat{3}),(\tilde{2},\hat{4})$

$(\hat{1},\tilde{3}),(\hat{1},\tilde{4}),(\hat{2},\tilde{3}),(\hat{2},\tilde{4})$

Quasimomenta:

$$\{\tilde{p}_1(x)\mid\hat{p}_1(x)\ ,\ \hat{p}_2(x)\mid\tilde{p}_2(x)\ ,\ \tilde{p}_3(x)\mid\hat{p}_3(x)\ ,\ \hat{p}_4(x)\mid\tilde{p}_4(x)\}$$

At the cuts/fermionic poles connecting sheets (ij) ″classical BAE″

$$\not{p}_i(x) - \not{p}_j(x) = 2\pi n_{ij}$$

Asymptotics for $x \to \infty$: connection becomes Noether currents, so p_i are related to global $\mathfrak{psu}(2,2|4)$ charges, in particular the classical energy E:

$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} \sim \frac{2\pi}{x\sqrt{\lambda}} \begin{pmatrix} +E - S_1 + S_2 \\ +E + S_1 - S_2 \\ -E - S_1 - S_2 \\ -E + S_1 + S_2 \\ +J_1 + J_2 - J_3 \\ +J_1 - J_2 + J_3 \\ -J_1 + J_2 + J_3 \\ -J_1 - J_2 - J_3 \end{pmatrix}$$

Other constraints:

- Poles at $x = \pm 1$: synchronized by Virasoro constraint.
- $x \to 1/x$ acts as automorphism of $\mathfrak{psu}(2,2|4)$:
  $p_{1,2,3,4}(1/x) \to -p_{2,1,4,3}(x)$

Quantizing the algebraic curve

Classical curve. Add small fluctuations, i.e. poles. Determine where they are localized and what the backreaction is,

$$p_i(x) \longrightarrow p_i(x) + \delta_n^{(ij)} p_i(x)$$

A single excitation of flavour (ij) and mode number n is defined by shifting the filling fractions $S_{ij} \longrightarrow S_{ij} + 1$. This fixes [Beisert, Freyhult], [Gromov, Vieira]

$$\delta_n^{(ij)} p_i(x) \sim \frac{\alpha(x_n^{ij})}{x - x_n^{ij}} \,, \qquad \alpha(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1}$$

The position of the fluctuation pole x_n^{ij} is determined by classical BAE

$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n_{ij}$$

Asymptotics at infinite are

$$\begin{pmatrix} \delta\hat{p}_1 \\ \delta\hat{p}_2 \\ \delta\hat{p}_3 \\ \delta\hat{p}_4 \\ \delta\tilde{p}_1 \\ \delta\tilde{p}_2 \\ \delta\tilde{p}_3 \\ \delta\tilde{p}_4 \end{pmatrix} \sim \frac{4\pi}{x\sqrt{\lambda}} \left(\begin{array}{ccc} +\delta\Delta/2 & +N_{\hat{1}\hat{4}} + N_{\hat{1}\hat{3}} & +N_{\hat{1}\tilde{3}} + N_{\hat{1}\tilde{4}} \\ +\delta\Delta/2 & +N_{\hat{2}\hat{3}} + N_{\hat{2}\hat{4}} & +N_{\hat{2}\tilde{4}} + N_{\hat{2}\tilde{3}} \\ -\delta\Delta/2 & -N_{\hat{2}\hat{3}} - N_{\hat{1}\hat{3}} & -N_{\tilde{1}\hat{3}} - N_{\tilde{2}\hat{3}} \\ -\delta\Delta/2 & -N_{\hat{1}\hat{4}} - N_{\hat{2}\hat{4}} & -N_{\tilde{2}\hat{4}} - N_{\tilde{1}\hat{4}} \\ \hline & -N_{\tilde{1}\tilde{4}} - N_{\tilde{1}\tilde{3}} & -N_{\tilde{1}\hat{3}} - N_{\tilde{1}\hat{4}} \\ & -N_{\tilde{2}\tilde{3}} - N_{\tilde{2}\tilde{4}} & -N_{\tilde{2}\hat{4}} - N_{\tilde{2}\hat{3}} \\ & +N_{\tilde{2}\tilde{3}} + N_{\tilde{1}\tilde{3}} & +N_{\hat{1}\tilde{3}} + N_{\hat{2}\tilde{3}} \\ & +N_{\tilde{1}\tilde{4}} + N_{\tilde{2}\tilde{4}} & +N_{\hat{2}\tilde{4}} + N_{\hat{1}\tilde{4}} \end{array} \right)$$

- Synchronized poles at $x = \pm 1$
- $x \to 1/x$
- close to cuts: $\delta p \sim \partial_x p$

General strategy to determine one-loop energy shift:

1. Make ansatz for δp_i that satisfies correct asymptotics and poles at x_n^{ij}, and $x = \pm 1$
2. Solve linear equations for undetermined constants and $\delta\Delta$

Efficient quantization of the algebraic curve

TBF[Gromov, SSN, Vieira]

The above procedure is already more efficient than semi-classical sigma-model approach à la Frolov-Tseytlin.

Can do better! Why? For more complicated solutions this is crucial to obtain exact, finite-size spectrum. E.g. GM as 2-cut solution.

Use $x \to 1/x$ symmetry, then from one frequency in S^3 and one in AdS_5

$$\tilde{\Omega}_n = \Omega_n^{\tilde{2}\tilde{3}}\,, \qquad \hat{\Omega}_n = \Omega_n^{\hat{2}\hat{3}}$$

we can determine all others as linear combinations.

NB: this is different from quantization in subsectors!

Let us consider an example.

Simple $S^3 \times \mathbb{R}$

Classical energy: $\kappa = \frac{E}{\sqrt{\lambda}} = \sqrt{\mathcal{J}^2 + m^2}$. J is spin, m is winding on S^3. The classical solution is determined by

$$\begin{aligned}
p_{\hat{1}} = p_{\hat{2}} = -p_{\hat{3}} = -p_{\hat{4}} &= +\frac{2\pi x}{x^2-1}\kappa \\
p_{\tilde{1}} &= +\frac{2\pi x}{x^2-1}\sqrt{\mathcal{J}^2 + \frac{m^2}{x^2}} \\
p_{\tilde{2}} &= +\frac{2\pi x}{x^2-1}\sqrt{\mathcal{J}^2 + m^2 x^2} - 2\pi m \\
p_{\tilde{3}} &= -\frac{2\pi x}{x^2-1}\sqrt{\mathcal{J}^2 + m^2 x^2} + 2\pi m \\
p_{\tilde{4}} &= -\frac{2\pi x}{x^2-1}\sqrt{\mathcal{J}^2 + \frac{m^2}{x^2}} \, .
\end{aligned}$$

Check: Asymptotics, poles, $x \to 1/x$.

Input fluctuations: $\Omega_n = \Omega(x_n)$

$$\tilde{\Omega}^{\tilde{2}\tilde{3}}(x_n) = \frac{2m + n_{\tilde{2}\tilde{3}}}{\kappa x_n} = \frac{2m + \frac{p_{\tilde{2}} - p_{\tilde{3}}}{2\pi}}{\kappa x_n} = \frac{2\sqrt{m^2 x_n^2 + \mathcal{J}^2}}{(x_n^2 - 1)\sqrt{m^2 + \mathcal{J}^2}}$$

$$\hat{\Omega}^{\hat{2}\hat{3}}(x_n) = \frac{2}{x_n^2 - 1}$$

Tracing back the $x \to 1/x$ symmetry of the quasi-momenta and δp, we can show that off-shell frequencies $\Omega(x)$, where $\Omega(x)|_{x=x_n} = \Omega_n$ satisfy

$$\Omega^{\tilde{1}\tilde{4}}(x) = -\Omega^{\tilde{2}\tilde{3}}(1/x) + \text{const.}\,, \qquad \Omega^{\hat{1}\hat{4}}(x) = -\Omega^{\hat{2}\hat{3}}(1/x) + \text{const.}$$

Remaining bosonic fluctuations are obtained by linear combinations of the off-shell frequencies

$$\Omega^{ij}(y) = \frac{1}{2}\left(\Omega^{ii'}(y) + \Omega^{j'j}(y)\right)$$

where

$$(\hat{1}, \hat{2}, \tilde{1}, \tilde{2}, \hat{3}, \hat{4}, \tilde{3}, \tilde{4})' = (\hat{4}, \hat{3}, \tilde{4}, \tilde{3}, \hat{2}, \hat{1}, \tilde{2}, \tilde{1})\,.$$

Applied to the $S^3 \times \mathbb{R}$ solution, we obtain all the well-known frequencies only in terms of the frequencies $\tilde{\Omega}$ and $\hat{\Omega}$

$$\Omega^{\tilde{1}\tilde{4}}(y) = -\Omega^{\tilde{2}\tilde{3}}(1/y) - 2\frac{\partial \mathcal{E}}{\partial \mathcal{J}}$$

$$\Omega^{\tilde{2}\tilde{4}}(y) = \Omega^{\tilde{1}\tilde{3}}(y) = \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) + \Omega^{\tilde{1}\tilde{4}}(y)\right) = \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) - \Omega^{\tilde{2}\tilde{3}}(1/y)\right) - \frac{\partial \mathcal{E}}{\partial \mathcal{J}}$$

$$\Omega^{\hat{1}\hat{4}}(y) = -\Omega^{\hat{2}\hat{3}}(1/y) - 2$$

$$\Omega^{\hat{2}\hat{4}}(y) = \Omega^{\hat{1}\hat{3}}(y) = \frac{1}{2}\left(\Omega^{\hat{2}\hat{3}}(y) + \Omega^{\hat{1}\hat{4}}(y)\right) = \frac{1}{2}\left(\Omega^{\hat{2}\hat{3}}(y) - \Omega^{\hat{2}\hat{3}}(1/y)\right) - 1$$

$$\Omega^{\hat{2}\tilde{4}}(y) = \Omega^{\tilde{1}\hat{3}}(y) = \frac{1}{2}\left(\Omega^{\hat{2}\hat{3}}(y) + \Omega^{\tilde{1}\tilde{4}}(y)\right) = \frac{1}{2}\left(\Omega^{\hat{2}\hat{3}}(y) - \Omega^{\tilde{2}\tilde{3}}(1/y)\right) - \frac{\partial \mathcal{E}}{\partial \mathcal{J}}$$

$$\Omega^{\tilde{2}\hat{4}}(y) = \Omega^{\hat{1}\tilde{3}}(y) = \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) + \Omega^{\hat{1}\hat{4}}(y)\right) = \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) - \Omega^{\hat{2}\hat{3}}(1/y)\right) - 1$$

$$\Omega^{\tilde{1}\hat{4}}(y) = \Omega^{\hat{1}\tilde{4}}(y) = \frac{1}{2}\left(\Omega^{\tilde{1}\tilde{4}}(y) + \Omega^{\hat{1}\hat{4}}(y)\right) = \frac{1}{2}\left(-\Omega^{\tilde{2}\tilde{3}}(1/y) - \Omega^{\hat{2}\hat{3}}(1/y)\right) - 1 - \frac{\partial \mathcal{E}}{\partial \mathcal{J}}$$

$$\Omega^{\hat{2}\hat{3}}(y) = \Omega^{\tilde{2}\hat{3}}(y) = \frac{1}{2}\left(\Omega^{\tilde{2}\tilde{3}}(y) + \Omega^{\hat{2}\hat{3}}(y)\right)$$

Summary: Efficient precision quantization

Finally the energy shift is computed by

- Solve for one $\tilde{\Omega}$ and one $\hat{\Omega}$
- Solve for the pole positions x_n^{ij} (trivial)

$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n_{ij}$$

- Evaluating the plug-in formula above for the off-shell frequencies with arbitrary polarization $\Omega^{ij}(x)$ and $x = x_n^{ij}$ (trivial)
- $\delta\Delta = \sum_{n\in\mathbb{Z}} \sum_{ij} (-1)^{F_{ij}} \Omega_n^{ij}$ (finished)

Application: GM as a 2-cut solution. Subleading exponential corrections.

In this way: get complete control over exponential terms at one-loop

$$\delta\Delta = \sum_{n\in\mathbb{Z}} \sum_{ij} (-1)^{F_{ij}} \Omega_n^{ij} = \oint dn cot(\pi n) \sum_{ij} (-1)^{F_{ij}} \Omega_n^{ij}$$

and expanding cotangent.

5. Wrapping up: Conclusions and Outlook

Summary:

- Finite-size effects are one crucial missing piece to be understood on the way to solve planar AdS/CFT
- 2-d field theory approach for the string: Lüscher formulas reproduce exponential terms in string energies
- Exponential terms at one-loop in α': efficient precision

Future directions:

- Application of Lüscher formulas to Konishi
- Beyond leading exponential corrections:
  From the algebraic curve: no problem at one-loop.
  Correct BAE to incorporate these
- TBA?
- Find more formal evidence for integrability

Thank you