Physics 12c, Problem set 1 Solutions

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[1] Molecules in a Box
(i) This problem is an example of the binary model system discussed in class. Instead of having
spins that are either up or down, we now have molecules that are either in the partition A or B.
Hence, it is straightforward to see that the multiplicity corresponding to having
$N_A$ molecules in partition A is given by $N_C^{N_A}$ (just as the multiplicity of a magnetic system with $k$ spins up is $N_C^k$.)

Because each molecule has two possible states, the total number of configurations is $2^N$ by the
fundamental counting principle. The probability $P(N_A)$ can then be obtained by dividing the
multiplicity by the total number of possible configurations:

$$P(N_A) = \frac{N_C^{N_A}}{2^N}$$

(ii) Again, we can take advantage of the fact that this problem is exactly analogous to the spin
system. In class, we saw that, by applying Stirling’s formula, a magnetic system with $k$ sites that
spin up is given by

$$(2\pi N)^{\frac{1}{2}} e^{-\frac{2s^2}{N}}$$

where $s = k - \frac{N}{2}$. All we have to do then is to replace ‘$s$’ in eq.(2) by $N_A - \frac{N}{2}$, which is exactly the
expression that you are asked to derive in the problem set.

(iii) The number of configurations that correspond to having $N_A$ molecules in partition A is, just as
before, $N_C^{N_A}$. However, what is different now is that the probability of each of those configurations
is no longer $\left(\frac{1}{2}\right)^N$, but rather $p^{N_A}(1 - p)^{N - N_A}$. ($N - N_A$ is the number of molecules in B, and the
chance of each molecule in B is given by $1 - p$; hence the factor $(1 - p)^{N - N_A}$.) The probability is
therefore

$$N_C^{N_A}(p)^{N_A}(1 - p)^{N - N_A}$$

(iv) To derive the large $N$ limit of the probability distribution, we first take the logarithm of eq(3).

$$\log(N_C^{N_A}(p)^{N_A}(1 - p)^{N - N_A}) = \log(\frac{N!}{N_A!N_B!}p^{N_A}(1 - p)^{N_B})$$

$$= \log(N!) - \log(N_A!) - \log(N_B) + N_A \log(p) + N_B \log(1 - p)$$

where $N_B = N - N_A$. 
Now apply Stirling’s formula \((\log(N!) = N \log(N) - N)\), and this simplifies to

\[
N \log(N) - N - N_A \log(N_A) + N_A - N_B \log(N_B) + N_B + N_A \log(p) + N_B \log(1 - p) \tag{6}
\]

\[
= \ N \log(N) - N_A \log(N_A) - N_B \log(N_B) + N_A \log(p) + N_B \log(1 - p) \tag{7}
\]

\[
= \ (N_A + N_B) \log(N) - N_A \log(N_A) - N_B \log(N_B) + N_A \log(p) + N_B \log(1 - p) \tag{8}
\]

\[
= \ -N_A \log \left( \frac{N_A}{Np} \right) - N_B \log \left( \frac{N_B}{N(1-p)} \right) \tag{9}
\]

At this point, it is convenient to introduce a new variable, \(s = N_A - Np\) (note that \(N_B = N(1-p) - s\)). Substituting \(s\) into eq(9), we have

\[
- N_A \log \left( 1 + \frac{s}{Np} \right) - N_B \log \left( 1 - \frac{s}{N(1-p)} \right) \tag{10}
\]

\[
= -(s + Np) \left( \frac{s}{Np} - \frac{1}{2} \left( \frac{s}{Np} \right)^2 \right) - (N(1-p) - s) \left( \frac{-s}{N(1-p)} - \frac{1}{2} \left( \frac{s}{N(1-p)} \right)^2 \right) \tag{11}
\]

\[
= - \left( \frac{s^2}{Np} + s \left( \frac{s}{Np} \right)^2 \frac{1}{2Np} \right) - \left( \frac{s^2}{N(1-p)} + s \left( \frac{s}{N(1-p)} \right)^2 \frac{1}{2N(1-p)} \right) \tag{12}
\]

\[
= - \frac{s^2}{2N} \left( \frac{1}{p} + \frac{1}{1-p} \right) \tag{13}
\]

\[
= - \frac{s^2}{2N} \left( \frac{1}{p(1-p)} \right) \tag{14}
\]

where in the second line, we Taylor expanded the logarithm. In going from the third to the fourth line, we threw away terms of order \(1/N^2\).

Exponentiating this expression, we see that the large \(N\) limit of the probability distribution is a Gaussian with a mean of \(Np\) and a variance of \(Np(1-p)\). These results are consistent with the central limit theorem.


(i) This problem can also be modelled as a binary model system. In this case, the ‘independent systems’ are the steps that the drunk takes. Again, there are two possible states for each step (left or right), just as the magnetic system or the molecules in a box. The variable \(N_A\) in [1], for example, assumes the role of the number of steps to the right in this problem.

It is obvious that if \(N\) is odd, the drunk can never return to the lamppost, as there is no way that displacement to the left can be cancelled by displacement to the right.

When \(N\) is even, we can apply the result we obtained in [1]i. Here, the condition that the drunk returns to the lamppost corresponds to the case in which \(N_A = N/2\). Hence, the probability \(P\) is

\[
P = NC_{N/2}(p)^{N/2}(1-p)^{N/2} \tag{15}
\]

\[
= NC_{N/2} \left( \frac{1}{2} \right)^N \tag{16}
\]

(ii) I will present here three methods to solve this problem.

The first solution: Assume that the first drunkard \(A\) ends up at \(s\) steps to the right (\(s\) could be negative). Imagine now that, instead of starting at the origin, the second drunkard \(B\) starts at \(s\) (where \(A\) ends up), and compute the probability that he walks back to the origin. Because of the
left/right symmetry of the problem, this is exactly the same probability as that of \( B \) starting at the origin and finishes at \( s \) (which is what we want)!!

So, the probability that they meet each other is identical to the probability that \( A \) walks to \( s \) and \( B \) walks back to the origin starting from \( s \) (and sum over all possible \( s \)'s). Now, because the steps taken by the two drunkards are independent of each other, this is really the same situation as having one drunkard walking back to the origin!

This implies that this problem is really identical to part a, except that the number of steps has doubled. The probability that we are after is therefore simply

\[
2N \binom{N}{N} \left( \frac{1}{2} \right)^{2N}
\]

\[ (17) \]

**The second solution:** Another way to tackle this problem is to consider the relative motion of the two drunkards. There are now three possibilities at each step: they can be two steps closer (with probability \( \frac{1}{4} \)); two steps further apart (with probability \( \frac{1}{4} \)); or equally far apart as before (with probability \( \frac{1}{2} \)).

This corresponds to a trinomial distribution (whereas part a is binomial). The probability that they meet, then, is simply

\[
\sum_{n_0} \frac{N!}{n_0! n_L! n_R!} \left( \frac{1}{2} \right)^{n_0} \left( \frac{1}{2} \right)^{n_L} \left( \frac{1}{2} \right)^{n_R}
\]

\[ (18) \]

\[
= \sum_{n_0} \frac{N!}{n_0!} \left( \frac{1}{2} \right)^{n_0} \left( \frac{1}{2} \right)^{(N-n_0)/2} \left( \frac{1}{2} \right)^{(N-n_0)/2}
\]

\[ (19) \]

where \( n_0, n_L, n_R \) are the number of steps corresponding respectively to a change in relative distance of 0, -2, and 2. Note that \( n_L = n_R = (N - n_0)/2 \). The sum runs over all even values \((0, 2, 4, \ldots)\) up to \( N \) if \( N \) is even and vice versa.

**The third solution:** The probability that the drunkard has taken \( R \) steps to the right after a total of \( N \) steps is given by

\[
N \binom{R}{N} \left( \frac{1}{2} \right)^{N}
\]

\[ (20) \]

In order for the other drunkard to end up at the same location, he must have taken exactly \( R \) steps to the right (though the ORDER in which he took those steps could be different). Thus the the probability that both end up at \( R \) steps to the right of the lamppost is given simply by squaring eq(20):

\[
\left( N \binom{R}{N} \left( \frac{1}{2} \right)^{N} \right)^2
\]

\[ (21) \]

To get the probability that they meet each other again at the end, we sum over all possible values of \( R \) (as different values of \( R \) correspond to disjoint outcomes):

\[
\sum_{R=0}^{N} \left( N \binom{R}{N} \left( \frac{1}{2} \right)^{N} \right)^2
\]

\[ (22) \]

[3] Large \( N \), Small \( p \)

(i) (a)
\[
\log(1-p)^N = N \log(1-p) \quad (23) \\
\approx N(-p) \quad (24) \\
\approx -\lambda \quad (25)
\]

Exponentiating this yields \(e^{-\lambda}\).

(b)

\[
\log\left(\frac{N!}{(N-N_A)!}\right) = \log(N!) - \log(N-N_A)! 
\approx N \log(N) - N - (N-N_A) \log(N-N_A) + (N-N_A) \quad (26) \\
\approx N \log(N) - (N-N_A) \log(N(1-N_A/N)) - N_A \quad (27) \\
\approx N \log(N) - (N-N_A) \log(N) - (N-N_A) \log(1-N_A/N) - N_A \quad (28) \\
\approx N_A \log(N) - (N-N_A)(-N_A/N) - N_A \quad (29) \\
\approx N_A \log(N) \quad (30)
\]

where we have applied Stirling’s formula in the second line, and taylor expanded the log in the fifth. Note that in the sixth line, all terms are dwarfed by the first one, which is of order \(\log(N)\).

Exponentiating the final expression gives \(N^{N_A}\).

(c) Recall that the probability distribution of [1] is

\[
N^C_{N_A}(p)^{N_A}(1-p)^{N-N_A} = \frac{N!}{N_A!(N-N_A)!} (p)^{N_A}(1-p)^{N-N_A} \quad (32)
\]

\[
= \left(\frac{N!}{(N-N_A)!}\right) (1-p)^{N-N_A}(p)^{N_A} \frac{1}{N_A!} \quad (33)
\]

Using (b), we can replace the first parenthesis of the second line by \(N^{N_A}\). This yields

\[
(N^{N_A})(1-p)^N(1-p)^{-N_A}(p)^{N_A} \frac{1}{N_A!} \quad (34)
\]

This further reduces with the use of (a):

\[
(N^{N_A})e^{-\lambda}(p)^{N_A}(1-p)^{-N_A}\frac{1}{N_A!} \approx \frac{(Np)^{N_A}}{N_A!}e^{-\lambda} \quad (35) \\
= \frac{(\lambda)^{N_A}}{N_A!}e^{-\lambda} \quad (36)
\]

The Poisson distribution approximates the binomial distribution when \(N\) is large and \(p\) is small. Note that, since \(p\) is very small, \((1-p)^{N_A} \approx 0\).

(ii) First note that the summation starts with \(n = 0\), not \(n = 1\).

It is straightforward to show that the Poisson distribution is normalized.

\[
\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \quad (37) \\
= e^{-\lambda} e^{\lambda} \quad (38) \\
= 1 \quad (39)
\]
Mean:
\[
\sum_{n=0}^{\infty} n \left( \frac{\lambda^n}{n!} e^{-\lambda} \right) = e^{-\lambda} \lambda \frac{d}{d\lambda} \sum_{n=0}^{\infty} \left( \frac{\lambda^n}{n!} \right) = e^{-\lambda} \lambda d\lambda (e^{\lambda}) = \lambda
\]

The mean is \( \lambda \).

**Standard Deviation:** First compute the second moment.
\[
\sum_{n=0}^{\infty} n^2 \left( \frac{\lambda^n}{n!} e^{-\lambda} \right) = e^{-\lambda} \left( \lambda \frac{d}{d\lambda} \right) \left( \lambda \frac{d}{d\lambda} \right) \sum_{n=0}^{\infty} \left( \frac{\lambda^n}{n!} \right) = e^{-\lambda} \lambda d\lambda (\lambda e^{\lambda}) = e^{-\lambda} e^{\lambda} (\lambda + \lambda^2) = \lambda + \lambda^2
\]

The standard deviation is \( \sqrt{(\lambda + \lambda^2) - \lambda^2} = \sqrt{\lambda} \).

(iii) (a) We can model this problem using the Poisson distribution. As mentioned, the Poisson distribution approximates the binomial distribution when \( N \) is large and \( p \) is small. Both of these conditions are satisfied in this problem: \( N \) is large because there are a lot of letters in the 600 pages; \( p \) is small because there are very few mistakes.

The value of \( \lambda \) is the number of errors per page; so \( \lambda = 1 \). The probability that a page contains no errors, therefore, is given by
\[
P_{\text{Poisson}}(0) = \frac{1^0}{0!} e^{-1} = e^{-1}
\]

(b) Instead of doing an infinite sum, we can compute the probability that a page contains at least 3 errors by subtracting the probability of having fewer than three errors from unity:
\[
1 - (P_{\text{Poisson}}(0) + P_{\text{Poisson}}(1) + P_{\text{Poisson}}(2)) = 1 - (e^{-1} + e^{-1} + e^{-1}/2) = 1 - \frac{5}{2} e^{-1}
\]