Ph12c HW 2 Solutions 4

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1 (a) Since
\[ g(U) = CU^{3N/2} \]  
we have
\[ \sigma(N, U) = \log(CU^{3N/2}) \]
\[ \Rightarrow \frac{1}{\tau} = (\frac{\partial \sigma}{\partial U})_N = \frac{3N}{2U} \]
\[ \Rightarrow U = \frac{3}{2}N\tau \]

(b) From (2) above we see that
\[ \frac{\partial^2 \sigma}{\partial U^2} = -\left(\frac{3N}{2}\right) \frac{1}{U^2} \]

2 For a system of \( N \) spins each of magnetic moment \( m \) in a magnetic field \( B \) with spin excess \( 2s \), the entropy is
\[ \sigma(s) \simeq \log g(N, 0) - \frac{2s^2}{N} \]
and total energy is
\[ U = -2smB \]
so that
\[ \sigma(U) = \log g(N, 0) - \frac{U^2}{2m^2B^2N} \]

Therefore
\[ \frac{1}{\tau} = \frac{\partial \sigma}{\partial U} = -\frac{U}{m^2B^2N} \]

Using (6) again, we obtain an expression relating \( \tau \) and \( s \)
\[ \frac{1}{\tau} = \frac{2s}{mBN} \]

so that the fractional magnetization at equilibrium is given by
\[ \frac{M}{Nm} = \frac{2s}{N} = \frac{mB}{\tau} \]
3 (a) Using the expression for the multiplicity function from chapter 1,
\[ g(N, n) = \frac{(N + n - 1)!}{n!(N - 1)!} \quad (11) \]
the entropy of a system of \( N \) quantum harmonic oscillators of total energy \( U = n\hbar\omega \) is
\[
\log(g) = \log((N + n - 1)! - \log(n! - \log(N - 1)!
= (N + n) \log(N + n) - (N + n) - n \log(n + n - N \log N + N
\]
We have used the Stirling approximation for both \( N \) and \( n \) and replaced \( N - 1 \) by \( N \). Thus
\[
\sigma(N, n) = N \log(1 + \frac{n}{N}) + n \log(1 + \frac{N}{n}) \quad (12)
\]
(b) Using \( U = n\hbar\omega \) in (12), we have
\[
\sigma(N, U) = N \log(1 + \frac{U}{\hbar\omega N}) + \frac{U}{\hbar\omega} \log(1 + \frac{N\hbar\omega}{U}) \quad (13)
\]
Differentiating with respect to \( U \),
\[
\frac{\partial \sigma}{\partial U} = \frac{N}{1 + \frac{U}{\hbar\omega N}} \log(1 + \frac{N\hbar\omega}{U}) - \frac{U}{\hbar\omega} \frac{1}{1 + \frac{N\hbar\omega}{U^2}} N\hbar\omega \quad (14)
\]
so that
\[
\frac{1}{\tau} = \frac{N}{U + \hbar\omega N} + \frac{1}{\hbar\omega} \log(1 + \frac{N\hbar\omega}{U}) - \frac{N}{N\hbar\omega + U}
\Rightarrow \frac{\hbar\omega}{\tau} = \log(1 + \frac{N\hbar\omega}{U}) \quad (15)
\]
This gives the desired expression for total energy at temperature \( \tau \):
\[
U = \frac{N\hbar\omega}{\exp(\frac{\hbar\omega}{\tau}) - 1} \quad (16)
\]
5 (a) Plugging in the values of \( \delta = 10^{11} \) and \( N_1 = N_2 = 10^{22} \) in the equation (17) of the example problem, ie.
\[
g_1(N_1, \delta)g_2(N_2, \delta) = (g_1g_2)_{max} \exp(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}) \quad (17)
\]
we have,
\[
\frac{g_1g_2}{(g_1g_2)_{max}} = \exp(-\frac{4 \times 10^{22}}{10^{22}}) = \exp(-4) = 1.83 \times 10^{-2} \quad (18)
\]
(b) First we note that summing $g_1(N_1, s_1)g_2(N_2, s-s_1)$ over $s_1$ is equivalent to integrating the expression $g_1(N_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_2 - \delta)$ over the corresponding range of values of $\delta$. Thus

$$\sum_{s_1} g_1(N_1, s_1)g_2(N_2, s-s_1) = \int \delta g_1(N_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_2 - \delta)d\delta$$

$$= (g_1g_2)_{\text{max}} \int \delta \exp(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2})d\delta$$

Thus the factor that we need to compute is $F = \int \delta \exp(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2})d\delta$ over the range of $\delta$.

Since $s_1$ takes values from $-\frac{N_1}{2}$ to $\frac{N_1}{2}$, and $\delta = s_1 - \hat{s}_1$, $\delta$ ranges from $-(\frac{N_1}{2} + \hat{s}_1)$ to $\frac{N_1}{2} - \hat{s}_1$. To compute $\hat{s}_1$ we make use of $\frac{\hat{s}_1}{N_1} = \frac{s}{N}$. In this case $N = 2N_1$ so that, $\hat{s}_1 = \frac{s}{2} = \frac{10^{50}}{2} = \frac{N_1}{200}$. This gives the range of $\delta$ as $-\frac{101N_1}{200}$ to $\frac{99N_1}{200}$ which is approximately $-\frac{N_1}{2}$ to $\frac{N_1}{2}$.

Therefore the required factor $F$ is

$$F = \int_{-\frac{N_1}{2}}^{\frac{N_1}{2}} \exp(-\frac{4\delta^2}{N_1})d\delta = 2 \int_{0}^{\frac{N_1}{2}} \exp(-\frac{4\delta^2}{N_1})d\delta$$

$$= \sqrt{\frac{N_1}{4}} \int_{0}^{\sqrt{N_1}} \exp(-x^2)dx$$

$$= \frac{\sqrt{\pi N_1}}{2} \text{erf}(\sqrt{N_1})$$

$$= \frac{\sqrt{\pi N_1}}{2} = 0.75 \times 10^{11} \quad (19)$$

(c) Entropy is given by $\sigma = \log(g) = \log[(g_1g_2)_{\text{max}}] + \log(F)$. Now,

$$(g_1g_2)_{\text{max}} = g_1(0)g_2(0) \exp(-\frac{s^2}{N})$$

$$= \frac{2}{\pi \sqrt{N_1 N_2}} 2^{N_1+N_2} \exp(-\frac{s^2}{N}) \quad (20)$$

Putting in the given values of $N_1, N_2, s$ we have,

$$\log[(g_1g_2)_{\text{max}}] = 2 \log(2)N_1 - 2 \frac{s^2}{N} - \frac{1}{2} \log(N_1 N_2) + \log\frac{2}{\pi}$$

$$= 10^{22} - 50 + \log\frac{2}{\pi} \simeq 10^{22} \quad (21)$$

On the other hand, the correction term is $\log(F) \simeq 11 \log(10) = 25.33$. 

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So the fractional error in entropy is

\[
\frac{\log(F)}{\log(g)} \sim \frac{\log(F)}{\log((g_1g_2)_{\text{max}})} \simeq 25 \times 10^{-22} \tag{22}
\]

Thus, even though \((g_1g_2)_{\text{max}}\) might not be a good approximation for \(\sum s_1 g_1(N_1, s_1) g_2(N_2, s - s_1)\) (as we saw in part (b)), the error induced in the entropy by this approximation is almost negligible. This is a consequence of the entropy being a logarithmic function of the multiplicity function \(g\) and therefore being additive in \(g\).