Ph12c HW 2 Solutions 4

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1 (a) Since

$$g(U) = CU^{3N/2} \tag{1}$$

we have

$$\begin{aligned} \sigma(N,U) &= \log(CU^{3N/2}) \\ \Rightarrow \frac{1}{\tau} = (\frac{\partial \sigma}{\partial U})_N &= \frac{3N}{2U} \qquad (2) \\ \Rightarrow U &= \frac{3}{2}N\tau \qquad (3) \end{aligned}$$

(b) From (2) above we see that

$$\frac{\partial^2 \sigma}{\partial U^2} = -(\frac{3N}{2})\frac{1}{U^2} \tag{4}$$

2 For a system of N spins each of magnetic moment m in a magnetic field B with spin excess $2s$, the entropy is

$$\sigma(s) \simeq \log g(N,0) - \frac{2s^2}{N} \tag{5}$$

and total energy is

$$U = -2smB \tag{6}$$

so that

$$\sigma(U) = \log g(N,0) - \frac{U^2}{2m^2B^2N} \tag{7}$$

Therefore

$$\frac{1}{\tau} = \frac{\partial \sigma}{\partial U} = -\frac{U}{m^2B^2N} \tag{8}$$

Using (6) again, we obtain an expression relating τ and s

$$\frac{1}{\tau} = \frac{2s}{mBN} \tag{9}$$

so that the fractional magnetization at equilibrium is given by

$$\frac{M}{Nm} = \frac{2s}{N} = \frac{mB}{\tau} \tag{10}$$

3 (a) Using the expression for the multiplicity function from chapter 1,

$$g(N,n) = \frac{(N+n-1)!}{n!(N-1)!} \tag{11}$$

the entropy of a system of N quantum harmonic oscillators of total energy $U = n\hbar\omega$ is

$$\begin{aligned}\log(g) &= \log(N+n-1)! - \log n! - \log(N-1)! \\ &= (N+n)\log(N+n) - (N+n) - n\log n + n - N\log N + N\end{aligned}$$

We have used the Stirling approximation for both N and n and replaced $N-1$ by N. Thus

$$\sigma(N,n) = N\log(1+\frac{n}{N}) + n\log(1+\frac{N}{n}) \tag{12}$$

(b) Using $U = n\hbar\omega$ in (12), we have

$$\sigma(N,U) = N\log(1+\frac{U}{\hbar\omega N}) + \frac{U}{\hbar\omega}\log(1+\frac{N\hbar\omega}{U}) \tag{13}$$

Differentiating with respect to U,

$$\frac{\partial\sigma}{\partial U} = \frac{N}{1+\frac{U}{\hbar\omega N}}(\frac{1}{\hbar\omega N}) + \frac{1}{\hbar\omega}\log(1+\frac{N\hbar\omega}{U}) - (\frac{U}{\hbar\omega})\frac{1}{1+\frac{N\hbar\omega}{U}}(\frac{N\hbar\omega}{U^2}) \tag{14}$$

so that

$$\begin{aligned}\frac{1}{\tau} &= \frac{N}{U+\hbar\omega N} + \frac{1}{\hbar\omega}\log(1+\frac{N\hbar\omega}{U}) - \frac{N}{N\hbar\omega+U} \\ \Rightarrow \frac{\hbar\omega}{\tau} &= \log(1+\frac{N\hbar\omega}{U})\end{aligned} \tag{15}$$

This gives the desired expression for total energy at temperature τ:

$$U = \frac{N\hbar\omega}{\exp(\frac{\hbar\omega}{\tau})-1} \tag{16}$$

5 (a) Plugging in the values of $\delta = 10^{11}$ and $N_1 = N_2 = 10^{22}$ in the equation(17) of the example problem, ie.

$$g_1(N_1,\hat{s_1}+\delta)g_2(N_2,\hat{s_2}-\delta) = (g_1g_2)_{\max}\exp(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}) \tag{17}$$

we have,

$$\frac{g_1g_2}{(g_1g_2)_{\max}} = \exp(-\frac{4\times10^{22}}{10^{22}}) = \exp(-4) = 1.83\times10^{-2} \tag{18}$$

(b) First we note that summing $g_1(N_1, s_1)g_2(N_2, s - s_1)$ over s_1 is equivalent to integrating the expression $g_1(N_1, \hat{s_1} + \delta)g_2(N_2, \hat{s_2} - \delta)$ over the corresponding range of values of δ. Thus

$$\begin{aligned}\sum_{s_1} g_1(N_1, s_1)g_2(N_2, s - s_1) &= \int_\delta g_1(N_1, \hat{s_1} + \delta)g_2(N_2, \hat{s_2} - \delta)d\delta \\ &= (g_1g_2)_{\max} \int_\delta \exp(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2})d\delta\end{aligned}$$

Thus the factor that we need to compute is $F = \int_\delta \exp(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2})d\delta$ over the range of δ.

Since s_1 takes values from $-\frac{N_1}{2}$ to $\frac{N_1}{2}$, and $\delta = s_1 - \hat{s_1}$, δ ranges from $-(\frac{N_1}{2} + \hat{s_1})$ to $\frac{N_1}{2} - \hat{s_1}$. To compute $\hat{s_1}$ we make use of $\frac{\hat{s_1}}{N_1} = \frac{s}{N}$. In this case $N = 2N_1$ so that, $\hat{s_1} = \frac{s}{2} = \frac{10^{20}}{2} = \frac{N_1}{200}$. This gives the range of δ as $-\frac{101N_1}{200}$ to $\frac{99N_1}{200}$ which is approximately $-\frac{N_1}{2}$ to $\frac{N_1}{2}$.

Therefore the required factor F is

$$\begin{aligned}F &= \int_{-\frac{N_1}{2}}^{\frac{N_1}{2}} \exp(-\frac{4\delta^2}{N_1})d\delta = 2\int_0^{\frac{N_1}{2}} \exp(-\frac{4\delta^2}{N_1})d\delta \\ &= \sqrt{N_1}\int_0^{\sqrt{N_1}} \exp(-x^2)dx \\ &= \frac{\sqrt{\pi N_1}}{2}\text{erf}(\sqrt{N_1}) \\ &= \frac{\sqrt{\pi N_1}}{2} = 0.75 \times 10^{11} \qquad (19)\end{aligned}$$

(c)Entropy is given by $\sigma = \log(g) = \log[(g_1g_2)_{\max}] + \log(F)$. Now,

$$\begin{aligned}(g_1g_2)_{\max} &= g_1(0)g_2(0)\exp(-\frac{s^2}{N}) \\ &= \frac{2}{\pi\sqrt{N_1N_2}}2^{N_1+N_2}\exp(-\frac{s^2}{N}) \qquad (20)\end{aligned}$$

Putting in the given values of N_1, N_2, s we have,

$$\begin{aligned}\log[(g_1g_2)_{\max}] &= 2\log(2)N_1 - 2\frac{s^2}{N} - \frac{1}{2}\log(N_1N_2) + \log(\frac{2}{\pi}) \\ &= 10^{22} - 50 + \log(\frac{2}{\pi}) \simeq 10^{22} \qquad (21)\end{aligned}$$

On the other hand, the correction term is $\log(F) \simeq 11\log(10) = 25.33$.

So the fractional error in entropy is

$$\frac{\log(F)}{\log(g)} \sim \frac{\log(F)}{\log[(g_1 g_2)_{\max}]} \simeq 25 \times 10^{-22} \tag{22}$$

Thus, even though $(g_1 g_2)_{\max}$ might not be a good approximation for $\sum_{s_1} g_1(N_1, s_1) g_2(N_2, s - s_1)$ (as we saw in part (b)), the error induced in the entropy by this approximation is almost negligible. This is a consequence of the entropy being a logarithmic function of the multiplicity function g and therefore being additive in g.