# Physics 12c, Problem set 3 Solutions

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**[1] Free energy of a harmonic oscillator**

The partition function is generally given by

$$Z = \sum_{s=0}^{\infty} e^{-\epsilon_s/\tau}. \tag{1}$$

For the harmonic oscillator, with energy levels $\epsilon_s = s\hbar\omega$, this becomes

$$Z = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau} = \frac{1}{1 - e^{-\hbar\omega/\tau}}. \tag{2}$$

The free energy is

$$F = -\tau \log Z = \tau \log(1 - e^{-\hbar\omega/\tau}). \tag{3}$$

(b)

The entropy can be computed from the free energy via

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V. \tag{4}$$

Substituting Eq. (3) into this expression for entropy, we obtain

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V = -\log(1 - e^{-\hbar\omega/\tau}) - \frac{\tau}{1 - e^{-\hbar\omega/\tau}}\left(-\frac{\hbar\omega}{\tau^2}\right)e^{-\hbar\omega/\tau} = \frac{\hbar\omega/\tau}{e^{\hbar\omega/\tau} - 1} - \log(1 - e^{-\hbar\omega/\tau}). \tag{5}$$

**[2] Energy Fluctuations**

In general, the expected value of energy is given by [see Eq. (3.12)]:

$$\langle\epsilon\rangle = U = \frac{\sum_s \epsilon_s e^{-\epsilon_s/\tau}}{Z} = \frac{\tau^2 \left(\frac{\partial Z}{\partial \tau}\right)_V}{Z}. \tag{6}$$

Taking a derivative of this result with respect to $\tau$, we obtain

$$\left(\frac{\partial U}{\partial \tau}\right)_V = -\frac{1}{Z^2}\left(\frac{\partial Z}{\partial \tau}\right)_V UZ + \frac{\sum_s \epsilon_s^2 e^{-\epsilon_s/\tau}}{\tau^2 Z} = -\frac{U^2}{\tau^2} + \frac{\langle\epsilon^2\rangle}{\tau^2}. \tag{7}$$

Multiplying by $\tau^2$, we obtain the desired result:

$$\langle\epsilon^2\rangle - (\langle\epsilon\rangle)^2 = \langle\epsilon^2\rangle - U^2 = \tau^2\left(\frac{\partial U}{\partial \tau}\right)_V. \tag{8}$$

**[3] Quantum Concentration**

We know from last term that the ground state energy of a particle of mass $M$ in a box is

$$E_0 = \frac{3\hbar^2\pi^2}{2ML^2} \tag{9}$$

Setting this to the $\tau$, we have

$$\begin{aligned} \frac{3\hbar^2\pi^2}{2ML^2} &= \tau & (10) \\ \frac{1}{L^2} &= \frac{2M\tau}{3\hbar^2\pi^2} & (11) \\ n_0 &= \left(\frac{1}{L^2}\right)^{3/2} = \left(\frac{2M\tau}{3\hbar^2\pi^2}\right)^{3/2} & (12) \end{aligned}$$

Comparing the final expression with equation (63) in the text, we see that $n_0$ is equal to $n_q$ within a factor of the order of unity.

**[4] Elasticity of polymers**

(a)

This system is once again very similar to the binary system model discussed in the lecture. Each link has two possible states: left and right. The expression $g(N,s) = \frac{N!}{(N/2+s)!(N/2-s)!}$ gives the multipliciy for the system to have an overall length of $2|s|$ to the right. However, for a particular value of $s$, $-s$ would also result in a chain of the same length. Taking that into account, the total entropy is given by

$$g(N,s)+g(N,-s) = \frac{N!}{(N/2+s)!(N/2+s)!} + \frac{N!}{(N/2+s)!(N/2+s)!} = 2\frac{N!}{(N/2+s)!(N/2+s)!} \tag{13}$$

(b)

Using equation (97) in the text, we can easily compute the entropy $\sigma$ of the system

$$\begin{aligned} \sigma &= \log(g(N,s)+g(N,-s)) & (14) \\ &= \log(2g(N,0)e^{-2s^2/N}) \\ &= \log(2g(N,0)) - 2s^2/N \\ &= \log(2g(N,0)) - l^2/2N\rho^2. \\ & & (15) \end{aligned}$$

In going from the first to the second line, we use the now familiar result that $g \approx g_0 e^{-2s^2/N}$. We obtain the last line by replacing $s$ in the third line by $\frac{l}{2\rho}$.

(c)

We calculate the external force by substituting the expression we got for the entropy into Equation (96) in Chapter 3.

$$\begin{aligned} f &= -\tau\left(\frac{\partial\sigma}{\partial l}\right)_{U,N} & (16) \\ &= \frac{l\tau}{N\rho^2} & (17) \end{aligned}$$