Ph12c HW 4 Solutions 4

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1 (a) Given, power radiated by the sun at a distance of $1AU$ (mean earth-sun distance) per unit area is $0.136 Js^{-1}cm^{-2}$. Therefore total power P, radiated by the sun is the solar constant times the area of a sphere of radius $= 1AU = 1.5 \times 10^{13} cm$.

$$P = 0.136 \times (4\pi(1.5 \times 10^{13})^2) = 3.85 \times 10^{26} Js^{-1} \quad (1)$$

(b) The Stefan-Boltzmann law states that the rate of energy emission per unit area is proportional to the fourth power of temperature(T), with $\sigma_B = 5.67 \times 10^{-12} Js^{-1}cm^{-2}K^{-4}$ as the proportionality constant. From (1), power radiated per unit area by the sun (J_U) is

$$J_U = \frac{P}{A_\odot} \quad (2)$$

where $A_\odot$ is the area of the sun. Using the given value of the radius of the sun, we find,

$$J_U = \frac{3.85 \times 10^{26}}{4\pi(7 \times 10^{10})^2} = 6.253 \times 10^3 Js^{-1}cm^{-2} \quad (3)$$

Using Stefan-Boltzmann law we get,

$$T^4 = \frac{6.253 \times 10^3}{5.67 \times 10^{-12}} = 1.1027 \times 10^{15} \quad (4)$$

Or $T = 5762.6K$. Thus, treating the sun as a black body, we find that its effective surface temperature is approximately $6000K$.

2 (a) For a self-gravitating homogenous sphere of mass M, radius R and density ρ, the gravitational self energy is obtained by integrating the gravitational potential energy over all particles in the sphere. Consider a spherical shell of radius r and thickness dr. The mass of this shell is $4\pi r^2 \rho$ and the mass of the volume interior to the shell is $\frac{4}{3}\pi r^3 \rho$ so that the gravitation potential energy of all points in the shell is $-\frac{G(\frac{4}{3}\pi r^3 \rho)(4\pi r^2 \rho)}{r}$. The gravitational self energy of the entire sphere is therefore

$$\begin{aligned} U &= -\int_0^R \frac{G(\frac{4}{3}\pi r^3\rho)(4\pi r^2\rho)}{r} \\ &= -\frac{16}{3}\pi^2 G\rho^2 \int_0^R r^4 dr \\ &= -\frac{16}{15}\pi^2\rho^2 G R^5 \\ &= -\frac{3GM^2}{5R} \end{aligned} \tag{5}$$

(b)Total thermal kinetic energy of the atoms in the sun is $\frac{3GM^2}{10R}$. Treating the particles in the sun as an ideal gas, we recall a result from chapter 3, that the average thermal kinetic energy is given by $\frac{3NK_BT}{2}$ where N is the number of particles in the sun and T is the temperature in the interior of the sun. Equating these two expressions we get an expression for T:

$$\begin{aligned} T &= \frac{GM^2}{5NK_BR} \\ &= 5.462 \times 10^6 K \end{aligned} \tag{6}$$

3 For a photon gas in 1 dimension, we have 2 transverse modes of propagation with mode frequencies of the form

$$\omega = \frac{n\pi v}{L} \tag{7}$$

where v is the velocity of transmission as given by the wave equation. The total energy of the photons in the line is, as given by equation (16):

$$U = \sum_n < \epsilon_n > = \sum_n \frac{\hbar\omega_n}{\exp(\hbar\omega_n/\tau) - 1} \tag{8}$$

Replacing the sum with an integral over the line element dn in the space of mode indices, we get, (with a factor of 2 to account for two polarizations)

$$U = 2\int_0^\infty \frac{\hbar\omega_n dn}{\exp(\beta\hbar\omega_n) - 1} \tag{9}$$

Note that this is just a 1-D integral, as opposed to the 3-D integral for a photon gas in a cavity. Defining $x = \beta\hbar\omega_n = (\beta\hbar n\pi v/L)$,

$$U = \frac{2L}{\beta^2\hbar v\pi}\int_0^\infty \frac{x dx}{e^x - 1} = \frac{\pi\tau^2 L}{3\hbar v} \tag{10}$$

The heat capacity is therefore

$$C_V = \frac{2\pi\tau L}{3\hbar v} \tag{11}$$

4 (a)Given $\tau V^{1/3}$ = a constant. Let V_0 be the volume of the universe when the radiation decoupled from matter and V denote the current volume of the universe. The $\tau_0 = 3000K$ and $\tau = 3K$ are the corresponding temperatures. Therefore

$$\frac{V_0}{V} = (\frac{\tau}{\tau_0})^3 = 10^{-9} \tag{12}$$

Approximately, $V \propto r^3$ the ratio of the corresponding radii is:

$$\frac{r_0}{r} = 10^{-3} \tag{13}$$

If the radius increased linearly with time, the decoupling happened at a time $t_0 = 10^{-3}t$ where t is the present age of the universe.

(b)Work done during an expansion from V_i to V_f is $W = \int_{V_i}^{V_f} PdV$. But from the thermodynamic identity derived in chapter 3(equation 34), $dU = \tau d\sigma - PdV = -PdV$ since the expansion is isentropic. Therefore work done is

$$\begin{aligned} W &= -\int_{U_i}^{U_f} dU \\ &= U_f - U_i \\ &= \frac{\pi^2}{15\hbar^3 c^3}[V_i\tau_i^4 - V_f\tau_f^4] \\ &= \frac{\pi^2}{15\hbar^3 c^3} V_i\tau_i^3(\tau_i - \tau_f) \end{aligned} \tag{14}$$

using the fact that $\tau V^{1/3}$ = a constant.