Physics 12c, Problem set 5 Solutions

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[1] Concentration Fluctuations.
(a)

$$\begin{aligned} < N^2 > &= \sum_s \frac{N^2 e^{(\mu N-\epsilon)/\tau}}{Z} && (1)\\ &= \frac{1}{Z}\sum_s \left(\tau\frac{\partial}{\partial\mu}\right)^2 e^{(\mu N-\epsilon)/\tau} && (2)\\ &= \frac{\tau^2}{Z}\frac{\partial^2}{\partial\mu^2}\sum_s e^{(\mu N-\epsilon)/\tau} && (3)\\ &= \frac{\tau^2}{Z}\frac{\partial^2}{\partial\mu^2} Z && (4)\end{aligned}$$

(b)
Using equation (80) and the result of (a), we have

$$\begin{aligned} < (\Delta N)^2 > &= < N^2 > - < N >^2 && (5)\\ &= \frac{\tau^2}{Z}\frac{\partial^2}{\partial\mu^2} Z - \left(\frac{\tau}{Z}\frac{\partial Z}{\partial\mu}\right)^2 && (6)\end{aligned}$$

On the other hand,

$$\begin{aligned} \tau\frac{\partial}{\partial\mu} < N > &= \tau\frac{\partial}{\partial\mu}\left(\frac{\tau}{Z}\frac{\partial Z}{\partial\mu}\right) && (7)\\ &= \tau^2\left(\frac{1}{Z}\frac{\partial^2 Z}{\partial\mu^2} - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial\mu}\right)^2\right) && (8)\end{aligned}$$

The last line is the same as eq(6). This establishes the equality.

[2] Ascent of sap in trees.
The intrinsic chemical potential of the water vapor is given by

$$\mu = \tau\log\left(\frac{n}{n_Q}\right) \quad (9)$$

At equilibrium, the chemical potential of the air that stands immediately above the pool of water and at the uppremost leaves are equal. The former is

$$\mu_{bottom} = \tau\log\left(\frac{n_0}{n_Q}\right), \quad (10)$$

while the latter also includes contribution from the gravitational potential:

$$\mu_{top} = \tau \log\left(\frac{rn_0}{n_Q}\right) + \frac{mgh}{N}. \tag{11}$$

Setting them equal, we have

$$\begin{aligned} \mu_{bottom} &= \mu_{top} && (12)\\ \tau \log\left(\frac{n_0}{n_Q}\right) &= \tau \log\left(\frac{rn_0}{n_Q}\right) + \frac{mgh}{N} && (13)\\ \tau \log(r) &= -\frac{mgh}{N} && (14)\\ h &= \frac{(-298K)(1.38 \times 10^{-23} J/K) \log(0.9)}{(9.8N/kg)(18)(1.67 \times 10^{-27} kg)} = 1471m && (15) \end{aligned}$$

[3] Energy of gas of extreme relativistic particles.

The partition function is

$$Z = \sum_s e^{-pc/\tau} = \sum_s e^{-hc/\lambda\tau} = \sum_s e^{-c\hbar n\pi/L\tau} \tag{16}$$

Turning the summation into an integral,

$$\begin{aligned} Z &\approx \frac{1}{8}\int_0^\infty (4\pi n^2 dn) e^{-c\hbar n\pi/L\tau} && (17)\\ &= \frac{\pi}{2}\int_0^\infty n^2 dn e^{-c\hbar n\pi/L\tau} && (18) \end{aligned}$$

Setting $x = \frac{\pi\hbar c}{L\tau} n$, the integral becomes

$$\begin{aligned} Z &= \frac{\pi}{2}\left(\frac{L\tau}{\pi\hbar c}\right)^3 \int_0^\infty dx x^2 e^{-x} && (19)\\ &= A\tau^3 && (20) \end{aligned}$$

where A is some constant independent of τ. We can now compute the energy from the partition function:

$$U = \frac{\tau^2}{Z}\frac{\partial Z}{\partial \tau} = \frac{\tau^2}{A\tau^3}(eA\tau^2) = 3\tau \tag{21}$$

[4] Time for a large fluctuation.

(a) We first calculate the number of particles of the system using the ideal gas law:

$$\begin{aligned} pV &= N\tau && (22)\\ N &= \frac{pV}{\tau} = \frac{(1.013 \times 10^5 Pa)(0.1 \times 10^{-3} m^3)}{(1.38 \times 10^{-23} J/K)(300K)} = 2.45 \times 10^{21} && (23) \end{aligned}$$

Dividing by the volume V, we get the density n:

$$n = \frac{2.45 \times 10^{-21}}{0.1 \times 10^{-3}} = 2.45 \times 10^{25} m^{-3} \tag{24}$$

Meanwhile, the quantum density n_Q is

$$n_Q = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2} = \left(\frac{(4\times 1.67\times 10^{-27}kg)(1.38\times 10^{-23}J/K)(300K)}{2\pi(1.05\times 10^{-34}Js)^2}\right)^{3/2} = 7.98\times 10^{30}m^{-3} \quad (25)$$

We now compute the entropy using the Sackur-Tetrode equation,

$$\sigma = N\left(\log\left(\frac{n_Q}{n}\right)+\frac{5}{2}\right) = (2.45\times 10^{21})\left(\log\left(\frac{7.98\times 10^{30}}{2.45\times 10^{25}}\right)+\frac{5}{2}\right) = 3.72\times 10^{22} \quad (26)$$

The number of accessible states g_1 can be calculated by exponentiating the entropy:

$$g_1 = e^{\sigma} = e^{3.72\times 10^{22}} \quad (27)$$

(b) We can easily see from the Sackur-Tetrode equation that, when the volume V increases by a factor β, the entropy σ increases by a factor of $N\log(\beta)$. Hence, if the gas is compressed to a volume of 0.05 litre, the entropy changes by $N\log(1/2) = -N\log(2) = -(2.45\times 10^{21})\log(2) = -1.70\times 10^{21}$.

The entropy (g_2) now becomes

$$g_2 = e^{3.72\times 10^{22}-1.70\times 10^{21}} = e^{3.55\times 10^{22}} \quad (28)$$

(c) The ratio is simply

$$\frac{g_2}{g_1} = e^{-1.70\times 10^{21}} = 10^{\log(e^{-1.70\times 10^{21}})} = 10^{-7.38\times 10^{20}} \quad (29)$$

(d) The number of collisions is simply the product of the number of particles N, the collision rate R, and the duration Δt(one year):

$$\begin{aligned} N_{collisions} &= NR\Delta t & (30)\\ &= (2.45\times 10^{21})(10^{10}s^{-1})(1yr) = 7.73\times 10^{38} & (31)\end{aligned}$$

(e) From part (a) and (b), we learned that for every $10^{7.38\times 10^{22}}$ configurations, there is one configuration in which all atoms are in half of the volume. If we assume that each collision corresponds to a new configuration, we expect that it would take roughly $10^{7.38\times 10^{22}}$ collisions to get to this particular microstate. Hence,

$$\begin{aligned} R\Delta t &= 10^{7.38\times 10^{22}} & (32)\\ \Delta t &= 10^{7.38\times 10^{22}}/(7.73\times 10^{28}yr^{-1}) & (33)\\ &\approx 10^{7.38\times 10^{22}}yr & (34)\end{aligned}$$

[5] **Gas of atoms with internal degree of freedom.**

(a) If we set the energy of the lower energy state to 0, the 'internal partition function' Z_{int} is given by

$$Z_{int} = 1 + e^{-\Delta/\tau} \quad (35)$$

Using equation (48) in the text, the chemical potential μ is

$$\mu = \tau\left(\log\left(\frac{n}{n_Q}\right) - \log(1+e^{-\Delta/\tau})\right) \quad (36)$$

(b) From (49) in text, the 'internal' free energy is

$$F_{int} = -N\tau \log(Z_{int}) = -N\tau \log(1 + e^{-\Delta/\tau}) \tag{37}$$

The total free energy F is then given by

$$F = N\tau \left(\log\left(\frac{n}{n_Q}\right) - 1 - \log(1 + e^{-\Delta/\tau}) \right) \tag{38}$$

(c) We first compute the 'internal' entropy:

$$\begin{aligned} \sigma_{int} &= -\left(\frac{\partial F_{int}}{\partial \tau}\right)_{N,V} && (39) \\ &= N\frac{\partial}{\partial \tau}(\log(1 + e^{-\Delta/\tau})) && (40) \\ &= N\left(\log(1 + e^{-\Delta/\tau}) + \frac{\Delta}{\tau}\frac{1}{e^{\Delta/\tau} + 1}\right) && (41) \end{aligned}$$

The total entropy is then

$$\sigma = N\left(\log\left(\frac{n_Q}{n}\right) + \frac{5}{2} + \log(1 + e^{-\Delta/\tau}) + \frac{\Delta}{\tau}\frac{1}{e^{\Delta/\tau} + 1}\right) \tag{42}$$

where the first two terms come from the Sackur-Tetrode equation. The last two terms are from eq(41).

(d) The pressure p is given by

$$p = -\left(\frac{\partial F}{\partial V}\right)_{\tau,N} = \frac{N\tau}{V} \tag{43}$$

This is simply the ideal gas law.

(e) We can compute U from F by performing a Legendre transformation:

$$\begin{aligned} U &= F + \tau\sigma && (44) \\ &= -N\tau\left(\log\left(\frac{n}{n_Q}\right) - 1 + \log(1 + e^{-\Delta/\tau})\right) + && (45) \\ &\quad \tau\left(N\left(\log\left(\frac{n_Q}{n}\right) + \frac{5}{2} + \log(1 + e^{-\Delta/\tau}) + \frac{\Delta}{\tau}\frac{1}{e^{\Delta/\tau} + 1}\right)\right) && (46) \\ &= F_0 + \frac{N\Delta}{e^{\Delta/\tau} + 1} && (47) \end{aligned}$$

where F_0 is the free energy without the internal degree of freedom. Note that the terms $N\tau\log(1 + e^{\Delta/\tau})$ cancel.

The heat capacity at constant pressure due to U_{int} is

$$C_{int} = \left(\frac{\partial u_{int}}{\partial \tau}\right)_p = \frac{N\Delta^2}{\tau^2}\frac{e^{\Delta/\tau}}{(e^{\Delta/\tau} + 1)^2} \tag{48}$$