Ph12c HW 2 Solutions 6

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1 (a) This is the same as problem 3(a) of Chapter-4, worked out in HW4.

(b) Recall that in the ground state, only the states up to the Fermi energy ϵ_F are occupied. Since the electrons are degenerate and non-relativistic, we can use equations (7) and (10) (in Chapter 7) for the Fermi energy and ground state kinetic energy of a degenerate Fermi gas of electrons:

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \tag{1}$$

(where m is the mass of an electron)

$$U_0 = \frac{3}{5}N\epsilon_F \tag{2}$$

Thus the ground state kinetic energy of the electrons in the white dwarf (of mass M and volume $V = 4/3\pi R^3$) is

$$\begin{aligned} U_0 &= \frac{3^{7/3}\pi^{2/3}}{4^{2/3}.5}\frac{\hbar^2}{2m}\frac{N^{5/3}}{R^2} \\ &\approx \frac{\hbar^2 N^{5/3}}{mR^2} \\ &\approx \frac{\hbar^2 M^{5/3}}{mM_H^{5/3}R^2} \end{aligned} \tag{3}$$

where we have left out the numerical factors in the second step and used the fact that the number of electrons is N = the number of ionized Hydrogen atoms = $\frac{M}{M_H}$ in the last step.

(c) Equating the expressions from parts (a) and (b) we have

$$\begin{aligned}\frac{GM^2}{R} &= \frac{\hbar^2 M^{5/3}}{mM_H^{5/3}R^2} \\ \Rightarrow M^{1/3}R &= \frac{\hbar^2}{GmM_H^{5/3}} \qquad (4)\\ &= 0.16667\times 10^{21} g^{1/3}cm \\ &\approx 10^{20}g^{1/3}cm \qquad (5)\end{aligned}$$

(d) Using the result of the last part, we get, $R = 1.323\times10^9 cm$ for a white dwarf of $M = 2\times10^{33}g$. Its density is therefore $\rho = 2.062\times10^5 gcm^{-3}$.

(e) In a neutron star it is the neutrons and protons that form a degenerate Fermi gas. Thus the results of the earlier parts of the problem go through except that m represents the mass of a neutron, instead of an electron. Since the neutron is a 1000 times heavier than the electron, the RHS of equation(4) get reduces by a factor of 1000. Thus $M^{1/3}R \approx 10^{17}g^{1/3}cm$.
If $M = 2\times10^{33}g$, $R \approx 10^6 cm = 10km$.

2 At low temperatures ($\tau < \tau_E$), the chemical potential of a non-interacting boson gas is close in energy to the ground state orbital, ie. $\mu \approx 0$ or $\lambda = e^{\frac{\mu}{\tau}} \approx 1$. Using equation(21) of chapter 7 and the expression for the density of states(equation 65), we have,
Energy of the spin zero boson gas is:

$$\begin{aligned}U &= \int_0^\infty \epsilon D(\epsilon) f(\epsilon,\tau) d\epsilon \\ &= \frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\epsilon^{3/2}d\epsilon}{e^{\epsilon/\tau}-1} \\ &= \frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2}\tau^{5/2}\int_0^\infty \frac{x^{3/2}dx}{e^x-1} \\ &= \frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2}\tau^{5/2} I(3/2) \qquad (6)\end{aligned}$$

where $\epsilon = x\tau$. Thus the temperature dependence of energy is $U \propto \tau^{5/2}$.
Specific heat at constant volume:

$$C_V = (\frac{\partial U}{\partial \tau})_V = (\frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2})\frac{5}{2}\tau^{3/2}I(3/2) \qquad (7)$$

The entropy and specific heat are related by $C_v = \tau(\frac{\partial\sigma}{\partial\tau})_V$. Thus the entropy is given by

$$\begin{aligned}\sigma &= \frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2} I(3/2)\int_0^\tau \tau^{1/2}d\tau \\ &= \frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2} I(3/2)\tau^{3/2} \qquad (8)\end{aligned}$$

3 First we show that the density of orbitals for a free electron in one dimension is $D(\epsilon) = \frac{L}{\pi}\left(\frac{2m}{\hbar^2\epsilon}\right)^{1/2}$ where L is the length of the line.
In one dimension, the quantum number of the highest occupied level is equal to half the total number of electrons ie. $2n_F = N$ (compare with equation(6) of Chapter 7 for 3-dimensions). Thus the Fermi energy is related to N as

$$\epsilon_F = \frac{\hbar^2}{8m}\left(\frac{\pi N}{L}\right)^2 \qquad (9)$$

where m is the electron mass. In other words, the number of free electron orbitals of energy less than or equal to ϵ is

$$N(\epsilon) = \sqrt{\frac{8m\epsilon}{\hbar^2}}\frac{L}{\pi} \qquad (10)$$

Since $D(\epsilon) = \frac{dN}{d\epsilon}$, we have,

$$D(\epsilon) = \left(\frac{2m}{\hbar^2\epsilon}\right)^{1/2}\frac{L}{\pi} \qquad (11)$$

For a one dimensional boson gas (of spin zero), we can use the same result, taking off a factor of 2 to account for the fact the zero spin of the bosons. Thus the number of particles in all excited orbitals is given by

$$\begin{aligned}N_e(\tau) &= \int_0^\infty D(\epsilon)f(\epsilon,\mu,\tau)d\epsilon \\ &= \frac{L}{\pi}\sqrt{\frac{m}{2\hbar^2}}\int_0^\infty \frac{d\epsilon}{\sqrt{\epsilon}(e^{\epsilon/\tau}-1)} \\ &= \frac{L}{\pi}\sqrt{\frac{m\tau}{2\hbar^2}}\int_0^\infty \frac{dx}{\sqrt{x}(e^x-1)} \qquad (12)\end{aligned}$$

where we have set $\lambda = 1$ is the second step and used $\epsilon = x\tau$ in the final step. It is easy to see that the definite integral diverges in the range 0 to 1 - $\sqrt{x}(e^x - 1) \leq (e-1)\sqrt{x}$ in the range $[0, 1]$ so that

$$\int_0^\infty \frac{dx}{\sqrt{x}(e^x-1)} \geq \int_0^\infty \frac{dx}{\sqrt{x}(e-1)} \qquad (13)$$

Since the integral on the RHS diverges, we see that the definite integral in (12) diverges. Thus a ground state condensate cannot form in a 1-D boson gas.

4 (a)Using equations (88) and (89) of chapter 7, we get an expression for the kinetic energy of the degenerate, relativistic gas of N electrons:

$$U_0 = \frac{3\hbar c N^{4/3}}{4R}\left(\frac{9\pi}{4}\right)^{1/3} \tag{14}$$

Equating this to the familiar expression for gravitational self energy we have

$$\frac{3\hbar c N^{4/3}}{4R}\left(\frac{9\pi}{4}\right)^{1/3} = \frac{3GM^2}{5R} \tag{15}$$

This allows us to predict the value of N in terms of the mass of the star

$$N = \frac{4}{(9\pi)^{1/4}}\left(\frac{GM^2}{5\hbar c}\right)^{3/4} \tag{16}$$

Assuming the star is entirely made up of hydrogen, we have, $M = Nm_H$ where m_H is the mass of a hydrogen atom. This gives

$$N = \sqrt{\frac{9\pi}{4}}\left(\frac{5\hbar c}{4Gm_H}\right)^{3/2} \tag{17}$$

(b)Substituting for the $\hbar$, c and m_H, the above expression gives the value $N = 8.4 \times 10^{57}$

5 The thermal average population of the lowest orbital is

$$< N(0) >= f(0,\tau) = \frac{1}{e^{-\mu/\tau} - 1} \tag{18}$$

and that of the orbital of energy ϵ is

$$< N(\epsilon) >= f(\epsilon,\tau) = \frac{1}{e^{(\epsilon-\mu)/\tau} - 1} \tag{19}$$

Given, $< N(0) >= 2 < N(\epsilon) >$ and $< N(0) > + < N(\epsilon) >= N$, so that,

$$\begin{aligned} \frac{1}{e^{-\mu/\tau} - 1} &= \frac{2N}{3} \\ \frac{1}{e^{(\epsilon-\mu)/\tau} - 1} &= \frac{N}{3} \end{aligned} \tag{20}$$

Inverting these to solve for τ, we have,

$$\begin{aligned} e^{-\frac{\mu}{\tau}} &= 1 + \frac{3}{2N} \\ e^{\frac{\epsilon-\mu}{\tau}} &= 1 + \frac{3}{N} \end{aligned} \tag{21}$$

or

$$\begin{aligned} -\frac{\mu}{\tau} &= \ln(1+\frac{3}{2N}) \\ \frac{\epsilon-\mu}{\tau} &= \ln(1+\frac{3}{N}) \\ \Rightarrow \frac{\epsilon}{\tau} &= \ln(1+\frac{3}{N}) - \ln(1+\frac{3}{2N}) \end{aligned} \tag{22}$$

Taylor expanding the log function using the fact that $N \gg 1$, we have,

$$\tau = \frac{2N\epsilon}{3} \tag{23}$$