Last Action it isn’t

We found classical equations of motion by demanding action be stationary. But in the classical path a minimum. There are always neighboring paths that measure the classical action. For example wiggly path below has almost same potential at except but much higher kinetic energy.

So the classical path is not a local maximum. But it isn’t always a local minimum either. As an example consider the harmonic oscillator

\[ L = \frac{1}{2} m x^2 - \frac{1}{2} k x^2 \]

with \( m = k = 1 \). Equation of motion:

\[ x'' + x = 0 \]

Solve \( x = x_0 \cos t \) for classical path.

At \( t = 0 \), \( x = x_0 \) and \( x = x_0 \) at \( t = 2\pi \). This path is that begins \( x \) and \( 0 \) coincide. Negligible path

\[ x(t) = (1 - \omega t) x_0 \cos t + \omega x_0 t \]

For some constant \( \omega \). Plug this in Lagrange and integrate to get action.
\[ S[x] = \int_0^{2\pi} \left[ \frac{(1-x^2)x^2}{2} \sin^2 t - \frac{1}{2} (\sqrt{1-x^2})^2 \cos^2 t \right] dt \]

\[ = -\pi x_0^2 x^2 \]

\( x=0 \) is classical path & any min & any max of action. So classical path is a saddle pt in this case not a local minimum.