Systems with N degrees of freedom

Two Particle Hilbert Space

Consider two particles moving in 1-dimension. System described classically by \((x_i, p_i)\) and \((x'_i, p'_i)\).

The rule for quantizing this system is to promote these classical variables to Hermitian operators \((X_i, P_i)\), \((X'_i, P'_i)\) obeying the canonical commutation relation

\[
[X_i, P_j] = i \hbar \delta_{ij}, \quad i, j \in \{1, 2\}
\]

\[
[X_i, X_j] = [P_i, P_j] = 0
\]

Usually one adopts coordinate basis consisting of ket

\(|x_i, x'_i \rangle\) and has simultaneous eigenstates of the
symmetry operator \(X_i, X'_i\)

\(X_i |x_i, x'_i \rangle = x_i |x_i, x'_i \rangle\)

\(X'_i |x_i, x'_i \rangle = x'_i |x_i, x'_i \rangle\)

and are normalized as

\[
\langle x'_i, x'_i | x_i, x_i \rangle = \delta(x'_i - x_i) \delta(x'_i - x_i)
\]

\[dx_i \text{ this gives us } \sqrt{x_i} \]

\(|\psi\rangle \rightarrow |x_i, x'_i \rangle |\psi\rangle = \Psi(x_i, x'_i)\)

\(x_i \rightarrow x_i\)

\(p_i \rightarrow \frac{-i\hbar}{\partial x_i}\)
We consider
\[ P(x_1, x_2) = |\Phi(x_1, x_2)|^2 \]
as probability density for finding particle 1 at \( x_1 \) and particle 2 at \( x_2 \), provided we normalize \( \Phi \) havint
\[ I = \langle \Psi | \Phi \rangle = \int |\langle x_1, x_2 | \Psi \rangle|^2 dx_1 dx_2 = \int P(x_1, x_2) dx_1 dx_2 \]

**V102 as a Direct Product Space**

There is another way to arrive at one Hilbert space. Consider any spin of 2 particles described classically by \( |x_1, p_1 \rangle \langle x_2, p_2| \). If we end up quantum defying
just of particle 1 we have operator \( \hat{X}_1 \), \( \hat{P}_1 \), satisfying
\[ [X_1, P_1] = i \hbar \]
The operators \( |x \rangle \) form a complete (orthonormal) basis. Operator \( X_1, P_1, S_z = S_z(x_1, p_1) \) act on this Hilbert space which we call \( V_1 \).

And particle 2 same deal.
\[ [X_2, P_2] = i \hbar \]

Equally, \( |x_2, p_2 \rangle \) form another basis too. We end up having \( X_2, P_2, S_z(x_2, p_2) \) act on the space.
Now we have a vector \( |x_1\rangle \) given \( x_1 \) spatial
2-level spin-1/2. Let's call smaller state

\( |x_{1'}\rangle \otimes |x_{1''}\rangle \rightarrow \) \( \text{partial 1 of} \ x_i \)

\( |x_{1'}\rangle \otimes |x_{1''}\rangle \) is called direct product of \( V_1 \) cell

\( \text{with} \ V_2 \). \( \text{Direct product of vectors in different}

\( \text{space} \ V_1 \) and \( V_2 \). Direct product is a linear

operation.

\[
(\langle x_1 | + \langle y_1 |) \otimes (\beta |z_2 \rangle
\]

\[
= \langle \beta | x_{1'} \rangle \otimes |x_{1''}\rangle + \langle \beta | x_{1'} \rangle \otimes |x_{1''}\rangle
\]

Set of all vectors \( |x_{1'}\rangle \otimes |x_{1''}\rangle \) form a basis for

a Hilbert space we call \( V_1 \otimes V_2 \). The direct

product of \( V_1 \) with \( V_2 \). Dimensionality of \( V_1 \otimes V_2
\)

is product of dimensionality of \( V_1 \) with

dimensionality of \( V_2 \).

Note coordinate basis \( |x_{1'}\rangle \otimes |x_{1''}\rangle \) is just one possible

basis. Could also use \( |f \rangle \otimes |h \rangle \) a more generic

\( |w, \rangle \otimes |v, \rangle \) for any Hermitian space \( \mathcal{H} \). Although

vectors defined for \( \text{spin} \) based space not any vector

space is of this form. In contrast

\[
|f \rangle = |x_{1'}\rangle \otimes |x_{1''}\rangle + |x_{1''}\rangle \otimes |x_{1'}\rangle
\]

cannot be used as \( |f \rangle \otimes |f \rangle \) when \( |f \rangle \in V_1 \)

and \( |f \rangle \in V_2 \). Inner product of \( |x_{1'}\rangle \otimes |x_{1''}\rangle

with \( |x_{1'}\rangle \otimes |x_{1''}\rangle \) is

\[
\langle x_{1'} | \otimes \langle x_{1''} | (| x_{1'} \rangle \otimes | x_{1''} \rangle) = \langle x_{1'} | x_{1'} \rangle \langle x_{1''} | x_{1''} \rangle
\]
Since any $\mathbf{x} \in \mathbb{W}_1 \mathbb{W}_2$ can be expressed in terms of $\mathbb{W}_1 \mathbb{W}_2$, let us define some product on both spaces. Now, define $X_1^{(1) \otimes (1)}$ analogously $X_i^{(1)}$ in the product space.


g\left(1^{(1)} \otimes 1^{(1)}\right) \mathbf{x}_1 \otimes \mathbf{x}_2 = \mathbf{x}_1 \otimes \mathbf{x}_2

g\left(1^{(1)} \otimes 1^{(1)}\right) \mathbf{x}_1 \otimes \mathbf{x}_2 = \mathbf{x}_1 \otimes \mathbf{x}_2

define the dual product of any two groups $\Gamma_1, \Gamma_2$ defined on $\Gamma_1 \otimes \Gamma_2$, to act on dual products but $\mathbb{W}_1 \otimes \mathbb{W}_2$, as

\Gamma_1 \otimes \Gamma_2 \mathbf{w}_1 \otimes \mathbf{w}_2 = \left(\Gamma_1 \mathbf{w}_1 \otimes \Gamma_2 \mathbf{w}_2\right)

So, $X_1^{(1) \otimes (2)} = X_1 \otimes I_2$

Following properties of dual products can be verified:

\[\text{iii] } \left[ X_1^{(1) \otimes 1}, X_1^{(1) \otimes 1}, X_1^{(1) \otimes 1} \right] = 0\]

\[\text{iii] } \left( X_1^{(1) \otimes 1} \right) \left( \Theta_1 \otimes \Lambda_2 \right) = \left( X_1 \Theta_1 \right) \otimes \left( \Lambda_2 \Lambda_2 \right)\]

\[\text{iii] } \left[ \Theta_1 \otimes 1, \Lambda_1 \otimes 1 \right] = \Gamma_1 \otimes I_2\]

\[\left[ X_1^{(1) \otimes 1}, X_1^{(1) \otimes 1}, X_1^{(1) \otimes 1} \right] = 0\]

\[\left[ X_1^{(1) \otimes 1}, X_1^{(1) \otimes 1}, X_1^{(1) \otimes 1} \right] = 0\]

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\[\left[ X_1^{(1) \otimes 1}, X_1^{(1) \otimes 1}, X_1^{(1) \otimes 1} \right] = 0\]
And the state $|\psi(\mathbf{x})\rangle = |\mathbf{x}_1\rangle \otimes |\mathbf{x}_2\rangle$

We will usually use $|\mathbf{x}\rangle$ notation but you should always remember direct product representation.

**Direct Product Span for coordinates has W.F. property**

Suppose $|\psi_1\rangle \otimes |\psi_2\rangle$ span $V_1 \otimes V_2$. General state will be

$$|\psi\rangle = \sum_{i,\nu} c_{i,\nu} |\psi_1^i\rangle \otimes |\psi_2^\nu\rangle$$

$$\langle x_1 | \otimes \langle x_2 | |\psi\rangle = \sum_{i,\nu} c_{i,\nu} \langle x_1 | \psi_1^i \rangle \langle x_2 | \psi_2^\nu \rangle$$

$$= \sum_{i,\nu} c_{i,\nu} \psi_1^i(x_1) \psi_2^\nu(x_2)$$

**Evolution of the Two-Particle State Vector**

$$i\hbar \frac{d}{dt} |\psi\rangle = \left[ -\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} + \frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^2} + V(x_1, x_2) \right] |\psi\rangle$$

$$= H |\psi\rangle$$

**Separable Hamiltonian $V(x_1, x_2) = V(x_1) + V(x_2)$**

$$H = H_1 + H_2$$

For a stationary state

$$i\hbar \frac{d}{dt} |\psi\rangle = \{-i\hbar \frac{d}{dt} \} |\psi\rangle = E |\psi\rangle$$

$$|\psi(t)\rangle = |E\rangle e^{-iEt/\hbar}$$
\[
\left[ H_1(x_1, p_1) + H_2(x_2, p_2) \right] |E\rangle = E |E\rangle
\]

Now, since \( \langle 1E | H_1, H_2 | 1E \rangle = 0 \), the ground state and excited state eigenvectors are

\[
|E_{1E}\rangle \otimes |E_{2E}\rangle
\]

\[
H_1 |E_{1E}\rangle = E_1 |E_{1E}\rangle
\]

\[
H_2 |E_{2E}\rangle = E_2 |E_{2E}\rangle
\]

\[
H |E\rangle = (E_1 + E_2) |E\rangle
\]

\[
|\Psi(t)\rangle = |E_{1E}\rangle e^{-iE_{1E}t} \otimes |E_{2E}\rangle e^{-iE_{2E}t}
\]

Now suppose we worked instead in coordinate basis

\[
\begin{bmatrix}
-\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} + V(x_1) \\
-\frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^2} + V(x_2)
\end{bmatrix}
|\Psi(x_1, x_2)\rangle
= E
|\Psi(x_1, x_2)\rangle
\]

\[
|\Psi(x_1, x_2)\rangle = \langle x_1, x_2 | E\rangle
\]

We consider this equation by the method of separation of variables. Assume

\[
|\Psi(x_1, x_2)\rangle = |\Psi_1(x_1)\rangle \otimes |\Psi_2(x_2)\rangle
\]

Putting in the L.H.S. and dividing by the quantum numbers

\[
\frac{1}{|\Psi_1(x_1)\rangle \langle \Psi_1(x_1)|}
\left[
-\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} + V(x_1)
\right]
|\Psi_1(x_1)\rangle
\]

\[
\left[
-\frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^2} + V(x_2)
\right] |\Psi_2(x_2)\rangle
\]

\[
+ \frac{1}{|\Psi_2(x_2)\rangle \langle \Psi_2(x_2)|}
\left[
-\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} + V(x_1)
\right] |\Psi_1(x_1)\rangle = E
\]
A function \( \psi_k \) with a fixed \( k \) equals a constant. So both the functions must separately be constant. Call the 2 constants \( E_1, E_2 \).

\[
\frac{1}{2m} \left[ -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi_k(x) = E_k \psi_k(x) = E_k \psi_k(x), \quad k = 1, 2
\]

\[
E = E_1 + E_2
\]

So we have

\[
\psi_k(x, x', t) = \psi_k(x, x', t) \exp \left( -i E_k t / \hbar \right)
\]

\[
= \psi_k(x, t) \exp \left( -i E_k t / \hbar \right) \exp \left( -i E_k t / \hbar \right)
\]

where \( \psi_k \), \( \psi_k \) are superpositions of corresponding 1-particle solutions, etc.

\( \Box \) Interacting Particles

\[
H = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + V(x_1, x_2), \quad V(x_1, x_2) = V(x_1 - x_2)
\]

\( \Box \) Classical

Then define center of mass coordinates
\[ x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \]

\[ x = x_1 - x_2 \]

\[ (m_1 + m_2)\ x_{cm} = m_1 x_1 + m_2 x_2 \]

\[ x_1 = (m_1 + m_2) = m_1 x + (m_1 + m_2)\ x_{cm} \]

\[ x_1 = \frac{m_2\ x + x_{cm}}{m_1 + m_2} \]

\[ x_1 = x_{cm} - \frac{m_1 x}{(m_1 + m_2)} \]

Now write Lagrange density in lens of the c

\[ L = \frac{1}{2} \left[ m_1 \dot{x}_1^2 + (m_2 \dot{x}_2^2 - V(x)) \right] \]

\[ = \frac{1}{2} m_1 \left( \frac{m_2}{m_1 + m_2} \dot{x}_1^2 + \frac{1}{2} m_1 \left( -m_1 \dot{x}_1^2 + x_{cm} \right)^2 \right) \]

\[ - V(x) \]

\[ = \frac{1}{2} \left( \frac{m_1 m_2 + m_2 m_1}{(m_1 + m_2)^2} \right) \dot{x}_1^2 + \frac{1}{2} \left( m_1 + m_2 \right) \dot{x}_{cm}^2 - V(x) \]

\[ \frac{1}{\hat{m}} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 m_2}{m_1 + m_2} \]

\[ m = \frac{m_1 m_2}{m_1 + m_2} \]
\[ H = \frac{p_m^2 + p_r^2}{2(M+m)} + V(x) \]

\[ p_m = \frac{\partial E}{\partial x} = (M+m) \frac{\partial x}{\partial x} \]

So again square \( \sigma \) or \( \eta \)

\[ \left[ x_m, p_m \right] = i \hbar \]

\[ \left[ x, \eta \right] = i \hbar \]

\[ H = \frac{p_m^2 + p_r^2 + V(x)}{2M} \]

\[ \psi_E(x_m, \eta) = \frac{e^{i p_m x_m / \hbar}}{(2\pi \hbar)^{1/2}} \Phi_{\text{rest}}(\eta) \]

\[ E = \frac{p_m^2}{2M} + E_{\text{rest}} \]

The non-helical dynamics is in \( E_{\text{rest}}(\eta) \), which is the energy eigenvalue for particles of mass \( m \) and a potential \( V(x) \).

**Generalizations**

**Mass particles in 1-dim - Straightforward.**

**Mass particles in n dimensions - Eigenstate \( |\mathbf{E}(x)\rangle = |x, y, z\rangle \)**
\[
\langle \xi | \bar{\xi} \rangle = \frac{1}{2^3} (\xi - \bar{\xi}) = \frac{1}{2} (x - x') \frac{1}{2} (y - y') \frac{1}{2} (z - z')
\]

**Identical Particles**

Call two particles identical if they are exact replicas of each other in any respect.

Suppose we have a system with 2 distinguishable particles (in 1-dimension) 1 and 2 and a measurement of position shows particle 1 to be at \( x = 0 \) and particle 2 to be at \( x = 6 \). After measurement

\[
1(4) = 1x = a, \quad x_2 = 6 \rangle = \langle a | 6 \rangle
\]

State obtained by unboosting \( 1(4) \)

\[
1(4) = 1 \alpha \rangle
\]

is different.

Now suppose particles are identical. Then two states at the same state \( 1(4)(a, b) \rangle \). Since one state is unidentical

\[
1(4)(a, b) = e^{2i\kappa} 1(4)(b, a) \rangle = e^{2i\kappa} 1(4)(a, b) \rangle
\]

\( \kappa = 0, \pi \).

But \( 1(4)(a) \rangle \) is an identical state and hence are not equivalent as \( 1(4)(a, b) \rangle \neq 1(4)(a, b) \rangle \)

Thus,

\[
1(4)(a, b) \rangle = \beta |a \rangle + \alpha |b \rangle
\]

\[
\sqrt{18^2 + 18^2}
\]
\[ 14(a, b)\rangle = 4 \cdot 14(b, a)\rangle \]

\[ \Rightarrow B = a \quad \text{and we can tell} \]

\[ 14(a, b)\rangle = \frac{1}{\sqrt{2}} [a b\rangle + b a\rangle] \quad \text{symmetric state} \]

and can also \[ 14(a, b)\rangle = -14(b, a)\rangle \]

\[ 14(b, a)\rangle = \frac{1}{\sqrt{2}} [a b\rangle - b a\rangle] \quad \text{antisymmetric state} \]

\text{Bosons + Fermions}

Now a given state of particles is either symmetric or antisymmetric. A symmetric state is one where the exchange of two particles maintains the same state. So, particles are divided into 2 types. Those with symmetric \( s \)'s are called bosons and those with antisymmetric \( s \)'s are called fermions.

Since we have two identical bosons \( x \) at one of them at \( x = 0 \) and at one of \( x = 6 \). Then know they are in state

\[ [a b\rangle + b a\rangle] \]

Since we have two identical fermions at one of them at \( x = 0 \) and at \( x = 6 \), then in state

\[ [a b\rangle - b a\rangle] \]
Now we have fermion states:

$$|w, w, A\rangle = \frac{|w, w, \uparrow\rangle - |w, w, \downarrow\rangle}{\sqrt{2}}$$

If \( w = 0 \), then \( |w, w, A\rangle = 0 \). So two fermions cannot be in same state!

All of this generalises to 8-dimensions. In 3-dimensions, a particle can have an internal angular momentum called spin. \( \hbar \) can take values \( 0, \frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2}, \ldots \). Particles with integer spin are bosons while those with half-integer spin are fermions. Beyond scope of this class to give explanation of this result called spin-statistical theorem.

Bosonic and Fermionic Hilbert spaces.

Space \( \mathcal{V}_{102} \) consists of all reducible \( |w, w, \rangle \) and all linear combinations of them, i.e. all reducible space of all symmetric bosonic states and all antisymmetric states. For each state \( |w, w, \rangle \) there is one bosonic state \( |w, w, s\rangle = \frac{1}{\sqrt{2}} (|w, w, \uparrow\rangle + |w, w, \downarrow\rangle) \) and similar for Fermionic \( |w, w, A\rangle = \frac{1}{\sqrt{2}} (|w, w, \uparrow\rangle - |w, w, \downarrow\rangle) \). Linear combination of symmetric or antisymmetric similar to antisymmetric. So they live in vector spaces. We can write

$$\mathcal{V}_{102} = \mathcal{V}_s \oplus \mathcal{V}_a$$

and any vector \( |w, w, \rangle \) can be decomposed into sum of \( \mathcal{V}_s \) and \( \mathcal{V}_a \)

$$|w, w\rangle = \frac{1}{\sqrt{2}} (|w, w, \uparrow\rangle + |w, w, A\rangle)$$
For basis elements of total $\hat{l}_z > 0$ we are associated with "quantum number" $m$, and each with $\ell_2$ is

\[ P_\ell(w, w_0) = \left| \langle w_0, w_0; \ell_2 | \hat{l}_z \rangle \right|^2 \]

\[ 1 = \langle \hat{l}_z | \hat{l}_z \rangle = \sum_{\text{dist}} \left| \langle w, w_0; \ell_2 | \hat{l}_z \rangle \right|^2 \]

\[ = \sum_{\text{dist}} P_\ell(w, w_0) \]

$\sum_{\text{dist}}$ is an element state.

\[ \sum = \sum_{w_0, \ell_{\text{min}}} \sum_{w, \ell_{\text{max}}} \]

so we are not counting $|w, w_0; \ell_2 \rangle$ and $|w, w_0; \ell_2 \rangle$ as different.

Now consider case where $D$ is scalar $X$ as are $w$ and $w_0$.

\[ P_\ell(x, x_0) = \left| \langle x, x_0; \ell_2 | \hat{l}_z \rangle \right|^2 \]

\[ 1 = \int \int \frac{P_\ell(x, x_0) \, dx_0 \, dx_2}{2} - \int \int \left| \langle x, x_0; \ell_2 | \hat{l}_z \rangle \right|^2 \, dx_0 \, dx_2 \]

Factor of $\frac{1}{2}$ makes for double counting done by $dx_0 \, dx_2$ integration.

Define out

\[ \chi_3(x, x_0) = \frac{1}{\sqrt{2}} \left( |x, x_0; \ell_2 \rangle + i |x_0, x; \ell_2 \rangle \right) \]

so that
\[ 1 = \iiint \frac{1}{\sqrt{\mathcal{L}(x, x')}} dx dx' \]

and then

\[ R_s(x, x') = 2/\mathcal{L}(x, x') \]

Note that

\[ \mathcal{L}(x, x') = \frac{1}{\sqrt{2}} \langle x_1, x_2 | \psi_s \rangle \]

\[ = \frac{1}{2} \left[ \langle x_1, x_2 | \psi_s \rangle + \langle x_2, x_1 | \psi_s \rangle \right] \]

\[ = \langle x_1, x_2 | \psi_s \rangle \quad \rightarrow \text{symmetric} \]

Now, with this

\[ 1 = \langle \psi_s | \psi_s \rangle = \iiint \frac{1}{\sqrt{\mathcal{L}(x, x')}} \mathcal{L}(x, x') dx dx' \]

\[ = \iiint \mathcal{L}(x, x') \langle x_1, x_2 | \psi_s \rangle \langle x_1, x_2 | \psi_s \rangle dx dx' \]

Now suppose as a corollary that we have two states in a 3-dim box with quantum number

- \( n = 3 \), \( n = 4 \) localized states

\[ | \psi_s \rangle = \frac{1}{\sqrt{2}} (13\downarrow + 14\uparrow) \]

\[ \mathcal{L}(x, x') = \frac{1}{\sqrt{2}} \langle x_1, x_2 | \psi_s \rangle \]

\[ = \frac{1}{2} \left( \langle x_1, x_1 | + \langle x_2, x_2 | \right) \left( \frac{13\downarrow + 14\uparrow}{\sqrt{2}} \right) \]
\[
\begin{align*}
\psi(x) &= \frac{1}{2\sqrt{2}} \left[ \psi_3(x_1) \psi_4(x_2) + \psi_3(x_2) \psi_4(x_1) + \psi_3(x_1) \psi_3(x_2) \right] \\
&= \frac{1}{2\sqrt{2}} \left[ \psi_3(x) \psi_4(x) + \psi_4(x) \psi_3(x) \right] \\
&= \langle x_1 x_2 | \psi \rangle
\end{align*}
\]

Recall that
\[
\psi_n(x) = \left( \frac{2}{L} \right)^{\frac{1}{2}} \sin \left( \frac{n\pi x}{L} \right)
\]

What about Fermions. Same idea:
\[
\begin{align*}
\langle \psi_1, \psi_2 | A \rangle &= \frac{1}{\sqrt{2}} \left( \langle \psi_1, \psi_2 | A \psi_1 \rangle - \langle \psi_1, \psi_2 | A \psi_2 \rangle \right) \\
\langle x_1 x_2 | \psi \rangle &= \frac{1}{\sqrt{2}} \langle x_1 x_2 | \psi \rangle \\
A(x_1, x_2) &= 2 |\psi(x_1, x_2)|^2 \\
1 &= \int \int A(x_1, x_2) \, dx_1 \, dx_2 = \int \int |\psi(x_1, x_2)|^2 \, dx_1 \, dx_2
\end{align*}
\]

Returning to our example. Suppose instead of being 2
particles, we put more fermions. Then state
\[
\langle \psi \rangle = \frac{1}{\sqrt{2}} \left( \langle \psi_1, \psi_2 \rangle - \langle \psi_3, \psi_4 \rangle \right)
\]
Wave function

\[ \psi(x, y) = \frac{1}{\sqrt{2}} \left( \psi_1(x) \psi_2(y) - \psi_2(x) \psi_1(y) \right) \]

In general from the states \( x_1, x_2 \)

\[ \psi(x_1, x_2) = \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_1(x_1) & \psi_2(x_1) \\ \psi_2(x_2) & \psi_1(x_2) \end{vmatrix} \]

Note: Equation in place 3, +

\[ \rho_{yx}(x_1, x_2) = \frac{1}{2} \left( \psi^*(x_1) \psi(x_2) + \psi^*(x_2) \psi(x_1) \right) \]

\[ = \frac{1}{\sqrt{2}} \left( \psi_1(x_1) \psi_2(x_2) \psi_1^*(x_2) + \psi_2(x_1) \psi_1(x_2) \psi_2^*(x_2) \right) \]

\[ + \frac{1}{\sqrt{2}} \left( \psi_1^*(x_1) \psi_2(x_2) \psi_1(x_2) + \psi_2^*(x_1) \psi_1(x_2) \psi_2(x_2) \right) \]

\[ \pm \frac{1}{\sqrt{2}} \left( \psi_1^*(x_1) \psi_2(x_2) + \psi_2^*(x_1) \psi_1(x_2) \right) \psi_3(x_2) + \text{h.c.} \]

With 48 mm hatch, parts 1-5 to part 1 in state 3 & part 2 in state 4, and zero in part 1 in state 2 for them and part 2 in state 3. Interference, barber, and liver...