System of \( N \) identical particles

System of three particles not quite enough to generate \( 6! \)
but consider \( 3 \) particles non interact in a triangle

\[
\begin{align*}
E_i &= \left( \frac{\pi^2}{2mL^2} \right) n^2
\end{align*}
\]

Particles are identical

Now \( \sigma = 3! \) product state to yield quantum number
\( \nu_1 \nu_2 \nu_3 \)

\( \left| 1 n_1 n_2 n_3 >, 1 n_1 n_3 n_2, 1 n_2 n_1 n_3 >, 1 n_2 n_3 n_1, \\
1 n_3 n_1 n_2, 1 n_3 n_2 n_1 > \right| \). Double reduction of identical
paticles gives us charge of \( \sigma = 3! \) single reduction can
give \( \pm 1 \). So particles are either down or up

\[
\begin{align*}
| n_1; n_2; n_3; S > &= \frac{1}{\sqrt{3!}} \left[ | n_1 n_2 n_3 > + | n_1 n_3 n_2 > + | n_2 n_1 n_3 > \right] \\
| n_1; n_2; n_3; A > &= \frac{1}{\sqrt{3!}} \left[ | n_1 n_2 n_3 > - | n_1 n_3 n_2 > + | n_2 n_1 n_3 > \right]
\end{align*}
\]

When \( X \) lives

\[
\Psi_A(x_1, x_2, x_3) = \frac{1}{\sqrt{5!}} \left< x_1, x_2, x_3 ; S/\sigma | \Psi_A \right>
\]

\[
= \left< x_1, x_2, x_3 | \Psi_A \right>
\]

\[
\int dx_1 dx_2 dx_3 | \Psi_A(x_1, x_2, x_3) |^2 = 1
\]
\[
\Psi_{n_1 n_2 n_3} (x_1, x_2, x_3; S/A) = \frac{1}{\sqrt{3}} \left[ \Psi_{n_1} (x_1) \Psi_{n_2} (x_2) \Psi_{n_3} (x_3) + \Psi_{n_1} (x_1) \Psi_{n_3} (x_2) \Psi_{n_2} (x_3) + \Psi_{n_2} (x_1) \Psi_{n_1} (x_2) \Psi_{n_3} (x_3) + \Psi_{n_2} (x_1) \Psi_{n_3} (x_2) \Psi_{n_1} (x_3) + \Psi_{n_3} (x_1) \Psi_{n_1} (x_2) \Psi_{n_2} (x_3) + \Psi_{n_3} (x_1) \Psi_{n_2} (x_2) \Psi_{n_1} (x_3) \right]
\]

The denominator may be written as a determinant.

\[
\Psi_{n_1 n_2 n_3} (x_1, x_2, x_3; S)
\]

\[
= \frac{1}{\sqrt{3}} \begin{vmatrix}
\psi_{n_1} (x_1) & \psi_{n_2} (x_1) & \psi_{n_3} (x_1) \\
\psi_{n_1} (x_2) & \psi_{n_2} (x_2) & \psi_{n_3} (x_2) \\
\psi_{n_1} (x_3) & \psi_{n_2} (x_3) & \psi_{n_3} (x_3)
\end{vmatrix}
\]
Symmetries in Classical Physics

I want to review a little about symmetry in classical physics before discussing them in QM. We say a transformation is

\[ q_i = q_i + \delta q_i (q,t) \]

\[ p_i = p_i + \delta p_i (q,t) \]

is a symmetry if it leaves the Hamiltonian invariant. Now a function \( g(q,t) \) is called the generator

of the above transformation if

\[ \delta q_i = \epsilon \dfrac{\partial g}{\partial q_i} \]

\[ \delta p_i = -\epsilon \dfrac{\partial g}{\partial q_i} \]

If the transformation generated by \( g \) is a symmetry then \( g \) is conserved.

\[ 0 = \delta H = \sum_i \dfrac{\partial H}{\partial q_i} \delta q_i + \dfrac{\partial H}{\partial p_i} \delta p_i = \epsilon \left( \dfrac{\partial H}{\partial q_i} \dfrac{\partial g}{\partial q_i} - \dfrac{\partial H}{\partial p_i} \dfrac{\partial g}{\partial p_i} \right) \]

\[ = \epsilon \{ H, g \}_{PB} \]

But recall \( \dfrac{dg}{dt} = \{ g, H \}_{PB} \) from dynamical variables so we have \( \delta H = 0 \Rightarrow \dfrac{dg}{dt} = 0 \)
Let's work on some example.

1. Suppose we have a system with Hamiltonian

\[ H = \sum_i \frac{\vec{p}_i^2}{2m_i} + \sum_{i,j} V(\vec{x}_i - \vec{x}_j) \]

This system has canonical coordinates, \( \vec{x}_i \) and momenta, \( \vec{p}_i \), and Hamiltonian is invariant under spatial translation.

\[ \vec{x}_i = \vec{x}_i + \vec{e} \]
\[ \vec{p}_i = \vec{p}_i \rightarrow \text{constant infinity} \]

So, take in the case for some quantum system.

\[ \vec{q} = \sum_i \vec{p}_i \]

For transformations that correspond to rotations, \( \vec{q} \) is angular momentum. Let's see this for a single particle on Z-axis.

\[ \delta x = -\epsilon y \quad \delta p = -\epsilon \vec{p}_y \]
\[ \delta y = \epsilon x \quad \delta p_x = \epsilon \vec{p}_x \]

\[ \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \text{call } \Theta = \epsilon \]

\[ \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x + \epsilon (-\vec{y}) \\ y + \epsilon \vec{x} \end{pmatrix} \]
Functun Hal guides this:

$$g = L_z = xpy - ypx$$

$$\frac{\partial g}{\partial x} = y \quad \frac{\partial g}{\partial y} = x$$

$$\frac{\partial g}{\partial x} = px \quad \frac{\partial g}{\partial y} = px$$

Symmetry & Their Consequences in Quantum Mechanics.

Consider a single particle in 1-dim. How do we define translational invariance? A particle in a state has well-defined energy & momentum. However, we expect momentum values to play the role of classical variables. So in translational invariance:

$$\langle x \rangle = \langle x \rangle + \epsilon$$

$$\langle p \rangle = \langle p \rangle$$

(Infinitesimal)

That is under translation each state $$|\psi\rangle$$ gets multiplied to a new state $$|\psi\rangle$$ such that

$$\langle \frac{\epsilon}{2} | x | \frac{\epsilon}{2} \rangle = \langle \frac{\epsilon}{2} | x | \frac{\epsilon}{2} \rangle + \epsilon$$

$$\langle \frac{\epsilon}{2} | p | \frac{\epsilon}{2} \rangle = \langle \frac{\epsilon}{2} | p | \frac{\epsilon}{2} \rangle$$

Now let define the translational gate

$$T(\epsilon|\psi\rangle = |\psi\rangle$$
There is also true

\[
\langle \psi | T^+(e) \times T(e) | \psi \rangle = \langle \psi | \times | \psi \rangle + \epsilon
\]

\[
\langle \psi | T^+(e) \rho T(e) | \psi \rangle = \langle \psi | \rho | \psi \rangle
\]

This is called the value part of new value

potential state is transformed by $T(e)$. Another

point of view, called passage is to say nothing

happens to the state vector, it is the operator $\rho$

that transform.

\[
X \rightarrow T^+ e X T(e)
\]

\[
\rho \rightarrow T^+ e \rho T(e)
\]

Such that

\[
T(e) \times T(e) = X + \epsilon I
\]

\[
T(e) \rho T(e) = \rho
\]

Moving partial one way is equivalent to moving

"environment" the other way.

Let first deal with value transformations

\[
T(e) | \psi \rangle = | \psi_e \rangle
\]

Here should $T(e)$ act if it is to correspond

to a translation. Answer is clear for popular

representations.
\[ T(e|\chi\rangle = |x + e\rangle \]

If particle is originally at \( x \) it ends up after transformation at \( x + e \). Once we know action of \( T(e) \) on this complete basis we can find it on any \( \chi \)

\[ |\psi\rangle = T(e)|\psi\rangle = \int dx \ T(e)|\chi\rangle \langle x|\psi\rangle \]

\[ = \int dx \ |x + e\rangle \langle x|\psi\rangle \]

\[ = \int dx' \ |x'\rangle \langle x'\psi|\rangle \quad x' = x + e \]

\[ = \int dx' \ |x'\rangle \psi(x' - e) = \int dx \ |x\rangle \psi(x - e) \]

So, apply action to \( \psi(x) \Rightarrow \psi(x - e) \)

Now let us write

\[ T(e) = \mathbb{1} - \frac{ie\mathcal{G}}{\hbar} \]

\[ \langle x|T(e)|\psi\rangle = \psi(x - e) \]

\[ \Rightarrow \psi(x) \left( -i\frac{\hbar}{\epsilon} \right) \langle x|\mathcal{G} |\psi\rangle = \psi(x) - \epsilon \frac{d}{dx} \psi(x) \]

\[ \psi(x) \left( -i\frac{\hbar}{\epsilon} \right) \langle x|\mathcal{G} |\psi\rangle = \psi(x) - \epsilon \frac{d}{dx} \psi(x) \]

Worked if \( \mathcal{G} = P \)

\[ T(e) = 1 - \frac{ie\mathcal{G}}{\hbar} \]

\[ T(e) = 1 - \frac{ie\mathcal{G}}{\hbar} \]
Note $T(E)$ is unitary to order $\epsilon$.

$$T^+(E) T(E) = \left(1 + \frac{i\epsilon P}{\hbar} \right) \left(1 - \frac{i\epsilon P}{\hbar} \right)$$

$$= 1 + O(\epsilon^2)$$

We want this since $\langle \Psi_e | \Psi_e \rangle = \langle \Psi_\psi \rangle$ implies symmetry ensures norm of states.

Translated as anything a an energy matrix

$$T(E) \dagger H T(E) = H$$

This makes sense that $\langle \Psi_\psi | H | \Psi_\psi \rangle = \langle \Psi_\psi | H | \Psi_\psi \rangle$

$$T^+(E) H T(E)$$

$$= H + \frac{i\epsilon [P, H]}{\hbar}$$

So for invariance $[L, H] = 0$. Note above

$$\langle L_\ell | H \rangle = \langle \ell | \psi \rangle = 0$$

momentum is conserved. Note what new $T(E)$ is unitary for an $S(\times \epsilon)$ that can be expanded in a power sum - $x, p$

$$T^+(E) S(\times \epsilon) T(E)$$

$$= S(\langle T^+(E) x T(E), \ T^+(E) p T(E) \rangle)$$
\[ H(x + \epsilon I, \rho) = H(x, \rho) \]

Time to do finite transformation:

**Finite Transformations**

Want operator that corresponds to point translation $a$.
Divide $a$ into $N$ parts of size $a/N$

\[ T(a/N) = 1 - \frac{ia}{hN} \]

\[ T(a) = \lim_{N \to \infty} \left[ T(a/N) \right]^N \]

\[ = e^{-i \rho a / h} = e^{-a(\partial / \partial x)} \]

Since \[ e^{-ax} = \lim_{N \to \infty} \left( 1 - \frac{ax}{N} \right)^N \]

So \[ \langle x | T(a) | \psi \rangle = \psi(x) - \frac{d}{dx} \psi(x) - \frac{d}{dx} \frac{d}{dx} \psi(x) \]

\[ = \psi(x-a) \]

Also note \[ T(a + b) = T(a) T(b) \].

Since \[ \langle T(a), H \rangle = 0 \]

\[ | \psi(t) \rangle = U(t) | \psi(0) \rangle \]

\[ T(a) | \psi(t) \rangle = T(a) | \psi(0) \rangle \]

\[ T(e_{1}) | \psi(t) \rangle = T(a) (U(t) | e_{1} \rangle) = U(t) (T(a) | e_{1} \rangle) \]