Theorem of quasi-lying, typically, matter

\[ \Omega \Lambda - \Lambda \Omega = \Lambda \Sigma \Lambda \]

is called the commutator of \( \Omega \) and \( \Lambda \). Typically, \( \Lambda \) doesn't vanish.

Useful commutator relations

\[ [\Sigma, \Lambda \theta] = \Lambda [\Sigma, \theta] + [\Sigma, \Lambda] \theta \]

\[ [\Lambda \theta, \theta] \Lambda [\Sigma, \theta] + [\Lambda, \theta] \Sigma \]

calculated \[ \Lambda [\Sigma, \theta] + [\Lambda, \theta] \Sigma \]

\[ = \Lambda [\Sigma, \theta] + 2 \Lambda \theta - 2 \theta \Sigma = -\Sigma \theta \]

The inverse of a quasi-quadratic \( \Sigma \) is denoted \( \Sigma^{-1} \)

and it satisfies

\[ \Sigma^{-1} \Sigma = \Sigma \Sigma^{-1} = 1 \]

If \( \det \Sigma \neq 0 \) then operator has inverse.

An quasi-quadratic has an inverse only if \( \Sigma V \Sigma = 0 \) implies \( V = 0 \);
then the quasi-quadratic is non-singular.

For a product of quasiquadratic \((\Lambda \Sigma)\)\(^T\) = \(\Sigma^{-1} \Lambda^{-1}\)

since

\[ (\Lambda \Sigma)\Lambda = \Sigma \Lambda \Lambda = \Omega \]

\[ \Lambda \Sigma = \Sigma \Lambda = \Omega \]

\[ \Lambda \Sigma^{-1} \Lambda = \Sigma \Lambda^{-1} \Lambda = \Omega \]

\[ \Lambda \Sigma \Lambda = \Sigma \Lambda \Lambda = \Omega \]
Suppose we are in a basis

\[ |V\rangle = \sum_i U_i |i\rangle \]

\[ \langle 2V | = \sum_i U_i' \langle i | \]

But \( \langle 2V | = \sum_i U_i \langle i | \psi \rangle \]

\[ = \sum_i U_i \langle i | j \rangle \langle j | \psi \rangle \]

\[ = \sum_i U_i \langle i | \psi \rangle \langle \psi | j \rangle \]

\[ U_i' = \sum_j \langle i | \psi \rangle \langle \psi | j \rangle U_j' = \sum_j D_{ij} U_j' \]

Note matrix elements of identity equal one

\[ \langle i | I | j \rangle = \delta_{ij} \quad \text{Kronecker delta} \]

Next usual article on basis

\[ |V\rangle = \sum_i |i\rangle \langle i | \psi \rangle \]

\[ = \sum_i (|i\rangle \langle i |) |V\rangle \]

So we can \( \sum_i |i\rangle \langle i | \) as a linear equal.
\[ I = \sum_i |i \rangle \langle i| = \sum_i P_i \]

\[ P_i = |i \rangle \langle i| \]

Note \( P_i P_j = \delta_{ij} P_j \)

\[ P_i |V \rangle = |i \rangle |V \rangle \]

\[ \langle V| P_i = v_i^* \langle i| \]

Now consider the product of two linear operators \( \Sigma \). \( (\Sigma A) |V \rangle = \Sigma (A |V \rangle) \).

\[ (\Sigma A) |i \rangle = \langle i | \Sigma A |i \rangle \]

\[ = \langle i | \Sigma |V \rangle |j \rangle \]

\[ = \sum_k \langle i | \Sigma |k \rangle \langle k | |V \rangle |j \rangle \]

\[ = \sum_k \delta_{ik} \lambda_{kj} \]

\[ \rightarrow \text{matrix multiplication} \]

We have that linear operators act on kets. You can also view them as acting on bra's

\[ (\langle V| \alpha + \langle V| \beta ) |V \rangle = \alpha \langle V| |V \rangle + \beta \langle V| |V \rangle \]
Now sometimes we write $\Omega |v\rangle = |Sv\rangle$. The corresponding bra vector $\langle \Omega |v\rangle$ which can write as some linear operator acting on $\langle v|$. That linear operator is the adjoint of $\Omega$.

$$\langle \Omega |v\rangle = \langle v|S^\dagger$$

What are its matrix elements

$$\langle i|S^\dagger |j\rangle = \langle \Omega i|j\rangle = \langle j|\Omega i\rangle^* = \langle j|\Omega |i\rangle^* = S_{ji}^*$$

$$(S^\dagger)_{ij} = S_{ji}^* \quad \text{(Note } (S^\dagger)^+ = S)$$

The adjoint of a product is the product of the adjoints in reverse.

$$(\Lambda S)^+ = \Lambda^+ S^+$$

Obvious from matrix multiplication but also

$$\langle \Omega \Lambda |v\rangle = \langle \Omega (\Lambda V)| = \langle \Lambda V |S^\dagger = \langle V|S^\dagger |\Lambda V$$

**Hermitean and Unitary Operators**

A linear operator $S$ is Hermitean if $S^\dagger = S$, and anti-Hermitean if $S^\dagger = -S$. We can decompose any such and into Hermitean + anti-Hermitean parts

$$S = \left[ \frac{S + S^\dagger}{2} \right] + \left[ \frac{S - S^\dagger}{2} \right]$$
Like decomposing a complex number into its real and imaginary parts. A unitary matrix $U$ satisfies

$$U U^* = U^* U = I$$

This is analogous to a complex number of unit modulus whose square $Z^2 = 1$. The special class of unitary operators is when they act on vectors they preserve the inner product between the

$$|v_i> = U|v_i> \quad |v'_i> = U|v'_i>$$

$$\langle v_i'|v_i> = \langle UV_i'|UV_i> = \langle v_i|U^*UU|v_i>$$

$$= \langle v_i|v_i>$$

If we look at any matrix of unitary operators, the columns can be thought of as vectors. Then we have

$$I = UU$$

$$\langle i|j> = \sum_k \langle i|U^*k> \langle k|j>$$

$$\delta_{ij} = \sum_k U^*_{ik} U_{kj}$$

Suppose we subject all vectors in our space to a unitary transformation $|\psi'\rangle = U|\psi\rangle$. Under this transformation the matrix elements of any linear operator $\Omega$ then

$$\langle v'|\Omega|v> \rightarrow \langle UV'|\Omega|UV>$$

$$= \langle v'|U^*\Omega UV>$$
So the same thing can be achieved by leaving the vector alone and transforming the operator

\[ S \rightarrow U^+SU \]

End are called an outer transformation and record

Consider some linear operator \( S \) acting on an arbitrary vector \( |IV\rangle \)

\[ S|IV\rangle = |IV\rangle \]

Usually \( |IV\rangle \) is very different from \( |IV\rangle \). But very operate \( S \) acts on vector called eigenvector for which action is a simple multiplication

\[ S|IV\rangle = \lambda |IV\rangle \]

We call \( |IV\rangle \) an eigenvalue of \( S \) and \( \lambda \) the eigenvalue. I want to remind you of the linear algebra and let you look an operator and find its eigenvalues & eigenvectors. Well, in some cases it's like linear. For which

\[ |IV\rangle = |IV\rangle \]

Small \( |IV\rangle \). For a regular \( \Pi_i = |i\rangle \langle i| \)

\( |IV\rangle \) are normalized but

\[ \Pi_i |i\rangle = |i\rangle \]

\[ \Pi_i |j\rangle = 0 |j\rangle \] for \( a |j\rangle \) orthogonal to \( |i\rangle \)
new does not mean you could. We can view $\Sigma |V\rangle = \omega |V\rangle$

\[
(\Sigma - \omega I) |V\rangle = 0
\]

So $\Sigma - \omega I$ has a non-trivial null space, but this means $\Sigma - \omega I$ is notinvertible and its determinant vanishes.

\[
\text{det}(\Sigma - \omega I) = 0
\]

This equation determines the eigenvalues $\omega$. We still need to find the eigenvectors.

\[
(\Sigma - \omega I) |V\rangle = 0
\]

\[
\sum_j \langle i | (\Sigma - \omega I) | j\rangle \langle j | V\rangle = 0
\]

\[
\sum_j (\Sigma_{ij} - \omega \delta_{ij}) U_j = 0
\]

Example:

\[
\Sigma = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
\]

This is actually a unitary operator $\Sigma \Sigma^* = I$.

\[
\text{det}(\Sigma - \omega I) = \begin{vmatrix}
1 - \omega & 0 & 0 \\
0 & -\omega & -1 \\
0 & 1 & -\omega
\end{vmatrix}
\]

\[
= (1-\omega) \begin{vmatrix}
-\omega & -1 \\
1 & -\omega
\end{vmatrix}
= (1-\omega)(\omega^2 + 1)
\]
The eigenvalues are $\omega = 1$, $\omega = i$, $\omega = -i$. Let us find eigenvectors.

(i) $\omega = 1$

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & -1 & -1 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

$v_2 + v_3 = 0 \Rightarrow v_2, v_3 = 0 \Rightarrow |\omega = 1 \rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$v_1 - v_3 = 0$

(ii) $\omega = i$

\[
\begin{bmatrix}
i & 0 & 0 \\
0 & -i & -1 \\
0 & 1 & -i
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\ i
\end{bmatrix}
\]

$v_1 = 0$, $-iv_2 - v_3 = 0 \Rightarrow v_2 = -iv_3 \neq \text{same vector}$

$v_2 = iv_3$

$|\omega = i \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

(iii) $\omega = -i$

Show $|\omega = -i \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$
An important property of the eigenvalues of a Hermitian matrix is that they are real.

\[
\lambda |w\rangle = w |w\rangle
\]

\[
\langle w | \lambda | w \rangle = w \langle w | w \rangle
\]

\[
\langle w | \lambda^+ | w \rangle = \langle w | \lambda | w \rangle^*
\]

\[
\lambda^2 = \lambda^+ \text{ subject to } (\lambda - \lambda^*) \langle w | w \rangle = 0 \Rightarrow \lambda = \lambda^*
\]

Suppose two eigenvalues of a Hermitian matrix are different.

\[
\lambda |w_i\rangle = w_i |w_i\rangle
\]

\[
\lambda |w_j\rangle = w_j |w_j\rangle
\]

\[
\langle w_j | \lambda | w_i \rangle = w_i \langle w_j | w_i \rangle
\]

\[
\lambda^+ = \lambda \quad \Rightarrow \quad \lambda |w_j \rangle = w_j |w_j \rangle
\]

\[
\Rightarrow \lambda w_i \text{ with } \lambda |w_j \rangle = 0
\]

So eigenbases corresponding to different eigenvalues are orthogonal. For there may be several ways to select eigenbases for the n real eigenvalues \( \lambda_{w_m} \), \( \lambda_{w_{m-1}} \), ..., \( \lambda_{w_1} \). Note that they are not necessarily orthogonal. They are a linearly independent subset of \( \mathbb{C}^n \). Perform Gram-Schmidt on the basis. So they can be chosen orthogonal. Hence any Hermitian matrix has an orthonormal basis of eigenvectors.
Well we know a lot about eigenvectors and eigenvalues for a Hermitian operator. Do not about unitary operator. Suppose

\[ U |U_i\rangle = |U_i\rangle \]

\[ U |U_j\rangle = |U_j\rangle \]

\[ N_{ij} = \langle U_j | U^+ U | U_i \rangle = U_i |U^*_j\rangle \]

\[ = \langle U_j | U_i \rangle(1 - U_i |U^*_j\rangle) = 0 \]

For \( i = j \) get \( 1 |U_i\rangle = 1 \) and for different.

If \( i \neq j \) \( \langle U_j | U_i \rangle = 0 \). Thus for the Hermitian case.

Consider a Hermitian operator \( \Sigma \) on \( H^\Omega(\mathbb{C}) \)

represented by a matrix in some orthonormal basis \( |U_1\rangle, \ldots, |U_n\rangle \) in some basis \( |U_1\rangle, \ldots, |U_n\rangle \) if \( i \neq j \) then in some vector that involves the change of basis

\[ |U_i\rangle = |U_j\rangle \]

Check its unitary and

\[ U^\dagger \Sigma U = \Sigma \]

is a diagonal matrix \( \Sigma \)

\[ \langle j | U^\dagger \Sigma U | i \rangle = \langle \omega_j | \Sigma | \omega_i \rangle = \delta_{ij} \]

Every Hermitian matrix can be diagonalized by a unitary matrix.