\[ H = \frac{1}{2m} (\vec{p} - e\vec{A})^2 + g\phi \]

Quantum Mechanics for a Single Particle in 1-Dim

1) The state of particle is represented by a vector in Hilbert space \( |\Psi(t)\rangle \)

(in classical mechanics specified by variables \( x(t), p(t) \), a pt in phase space)

2) The independent variable \( x \) and \( p \) of the Hamiltonian formulation of classical mechanics are represented by Hermitian operators \( X \) and \( P \) with matrix elements in eigenbasis of \( x \):

\[ \langle x' | X | x \rangle = x \delta(x - x') \]

\[ \langle x | P | x' \rangle = -i\hbar \delta'(x - x') \]

The operators corresponding to dependent variables \( \Omega (x, p) \) are given by Hermitian operators

\[ \Omega (x, p) = \omega (x' \rightarrow x, p \rightarrow P) \]

(in classical mechanics every dynamical variable \( \omega \) is a function of \( x \) and \( p \): \( \omega = \omega (x, p) \))
3) If a particle is in state \( |4\rangle \) measurement of the
variable (corresponding to \( S_2 \)) will yield one of its eigenvalues with probability
\[
P(\omega) \propto |\langle W|\omega\rangle|^2
\]
which is non-degenerate.

The state of the system will change from \( |4\rangle \) to \( \{0\} \)
as a result of the measurement.

(in classical mechanics measurement of \( S_2 \) yields \( d(x,p) \)
while system doesn't change)

4) The state obeys time-like dependence \( |4(t)\rangle \)
given by the Schrödinger equation
\[
i\hbar \frac{d}{dt} |4(t)\rangle = H |4(t)\rangle
\]
where
\[
H(x,p) = H(x,\dot{x}, p, \dot{p} = \frac{p}{m})
\]

Classical mechanics is not available \( x(t), \dot{x}, \dot{p} \), change
according to
\[
\dot{x} = \frac{\partial H}{\partial \dot{x}}, \quad \dot{p} = -\frac{\partial H}{\partial x}
\]

Suppose particle is in state \( |4\rangle \) and we want to measure \( \omega(x, p) \). In quantum mechanics \( \omega(x, p) = S_2 = \omega(x, p) \).
Firstly find the 0.n eigenvalue \( \langle W | \omega \rangle \) of \( S_2 \).
Expand \( |4\rangle \) in the basis
\[
|4\rangle = \sum_i \omega_i |\omega_i\rangle
\]

Probability \( P(\omega) \) that result \( \omega \) is obtained is
\[
P(\omega) \propto |\langle W | \omega \rangle|^2 = |\langle 4 | \omega \rangle |^2 = |\langle 4 | \omega | 4 \rangle|^2
\]
\[
= |\langle \omega_4 | \omega | 4 \rangle|^2
\]
Theory only makes probabilistic statements, hence normalization yields eigenvalues so only result of a measurement possible is an eigenvalue. To get normalized probability

\[ P(\psi_i) = \frac{|\langle \psi_i | \psi \rangle|^2}{\sum_i |\langle \psi_i | \psi \rangle|^2} \]

\[ = \frac{|\langle \psi_i | \psi \rangle|^2}{\langle \psi | \psi \rangle} \]

If we used normalized \( \psi \)

\[ |\psi'\rangle = \frac{|\psi\rangle}{\langle \psi | \psi \rangle} \]

then \( P(\psi_i) = |\langle \psi_i | \psi' \rangle|^2 \).

Note also our states \( \psi \) are not normalized, but eigenvalues \( \lambda \), \( \langle x | x' \rangle = \delta(x-x') \). Talk more about this later.

If two states are superposed

\[ |\psi\rangle = \alpha |\psi_1\rangle + \beta |\psi_2\rangle \]

\[ \frac{1}{\sqrt{1+\beta^2}} \]

denoted as a result of measuring \( \delta \)-wave

\[ \lambda_1 \approx (\psi_1 + \beta \psi_2), \quad \lambda_2 \approx (\psi_1 - \beta \psi_2) \]

Eigenstates
We had fun rules for quantum mechanics but there are some problems that could arise when you try to apply them.

Complication (i) The recipe $\Sigma = \omega (x \cdot x, p \cdot p)$ is ambiguous. For example, $\omega = xp$ we don't know if $\Sigma = xp$ or $p^2$ since $px = xp$ classically. In this case one would use the symmetric sum $\Sigma = (xp + px)/2$. This makes $\Sigma$ Hermitian. I don't know of a general prescription.

Complication (ii) The operator $\Sigma$ is degenerate. Let us say $\omega_1 = \omega_2 = \omega$. What is $p(\omega)$ in this case? Select some orthonormal $|w, 1\rangle, |w, 2\rangle$ for the eigenstates $|w\rangle$ with eigenvalues $\omega$. Project on $\omega$ in the subspace

$$P(\omega) = |w, 1\rangle \langle w, 1| + |w, 2\rangle \langle w, 2|$$

$$P(\omega) = \langle \psi | P_{\omega} | \psi \rangle = \langle P_{\omega} \psi | P_{\omega} \psi \rangle$$

$$P(\omega) = \langle \psi | P_{\omega} | \psi \rangle$$

is more general result

$$\text{Collapsing} |\psi\rangle \xrightarrow{\text{measurement}} P_\omega |\psi\rangle \xrightarrow{\text{measurement}} \langle \psi | P_{\omega} | \psi \rangle$$

Complication (iii) The eigenvalue spectrum of $\Sigma$ is continuous. In this case one expands

$$|\psi\rangle = \int |\omega\rangle \langle \omega | \psi \rangle d\omega$$

Note

$$\langle \omega | \psi \rangle = \int \langle \omega | \omega \rangle \langle \omega | \psi \rangle d\omega$$

$$\langle \psi | \omega \rangle = \int \langle \psi | \omega \rangle \langle \psi | \omega \rangle d\omega$$

$$\Rightarrow \langle \psi | \omega \rangle = \delta (\omega - \omega')$$
\( \psi(w) = \langle \psi|\psi \rangle \) is called the wave function for \( \psi \) in \( w \)-space. This is not a new field since it got away of generalizing \( \psi \). \( \psi(w) \) is called the probability amplitude for finding the particle with \( w = w \).

Note: \( \psi(w) = |\langle \psi|w \rangle|^2 \) is not the probability of finding the particle and \( \Delta w = \Delta w \), each such value of \( w \) can be assigned only an unphysical probability. One interprets \( \psi(w) \) as a probability density.

\[
\int |\psi(w)|^2 dw = \int |\langle \psi|w \rangle|^4 dw \\
= \int |\langle \psi|w \rangle| \langle \psi|w \rangle dw \\
= \langle \psi|\psi \rangle = 1.
\]

If \( |\langle \psi|w \rangle|^2 = \delta(w) \) the only smooth normalized wave function can be the normalized \( \psi(w) \) itself integrated as a relative probability density.

Important: the spectrum of \( \psi \) upwards belonging to position \( x \). \( \psi(x) \) is usually called wave function. \( |\psi(x)|^2 \) is probability density for finding particle at ladder \( x \).

Complication: The quantum variable \( \Delta x \) has no classical analog. We just do an best and get helped by exprience. Eg. spin of particle.

**Testing Quantum Mechanics**

Quantum mechanics only makes probabilistic predictions or statistical predictions. Suppose we prepare our particle in a state \( \psi \). All we know
some variable \( A \) immediately after preparation

\[
\hat{\text{w}} = \frac{1}{\sqrt{3}} (\hat{\omega}_1 + \sqrt{2} \hat{\omega}_2 + 0.1 \hat{\omega}_3)
\]

The very predict \( d_1, d_2 \) will be obtained with probability \( \frac{1}{3} \) or \( \frac{2}{3} \). Suppose we get \( d_1 \) from our measurement. That doesn't lead anywhere. Must repeat experiment a large number \( N \) times to verify that \( \hat{\omega}_1 \) time get \( d_1 \) \( + \sqrt{2} \hat{\omega}_2 \) get \( d_2 \). Never could repeat experiment with this particle after measurement since changed a \( t \). Must start again. So talk of quantum ensemble of \( N \) identical systems all in same state \( \hat{\omega} \) or an ensemble of \( N \) molecules in same state. Do measurement on each member of ensemble a

**Expectation Value:**

Given an ensemble of \( N \) particles in a state \( \ket{\psi} \), quantum mechanics allows us to predict the fraction with yield a value \( \omega \) if variable \( \hat{\omega} \) is measured. This encodes solving eigenvalue problem for \( \hat{\omega} \). Suppose one doesn't need the detailed information but just wants the average value of \( \hat{\omega} \) over the ensemble. This is called the expected value of \( \hat{\omega} \) denoted \( \langle \omega \rangle \)

\[
\langle \omega \rangle = \frac{1}{N} \sum_i \langle \psi_i | \hat{\omega} | \psi_i \rangle = \frac{1}{N} \langle \psi | \hat{\omega} | \psi \rangle
\]

\[
= \langle \hat{\psi} | \hat{\omega} | \psi \rangle = \langle \hat{\psi} | \sum_i \langle \psi_i | \hat{\omega} | \psi_i \rangle \psi_i \rangle
\]

\[
= \frac{1}{N} \langle \sum_i | \psi_i \rangle | \hat{\omega} | \langle \psi_i | \sum_i | \psi_i \rangle \rangle = \langle \hat{\psi} | \hat{\omega} | \psi \rangle = 
\]
Well we computed the average value of $\omega$ measured, what about the standard deviation for this average:

$$\Delta \omega = \sqrt{\langle (\omega - \langle \omega \rangle)^2 \rangle}$$

$$\langle \Delta \omega \rangle^2 = \sum_i P_i(\omega_i) (\omega_i - \langle \omega \rangle)^2$$

$$\Delta \omega = \sqrt{\langle (\omega - \langle \omega \rangle)^2 \rangle}$$

Note the above results hold even if spectrum is continuous.

Then

$$\langle \Delta \omega \rangle^2 = \int P(\omega) (\omega - \langle \omega \rangle)^2 \, d\omega$$

$$= \int \langle \omega \rangle^2/\psi^2 (\omega - \langle \omega \rangle)^2 \, d\omega$$

$$= \int d\omega \langle \psi | \omega \times \psi | \omega \rangle (\omega - \langle \omega \rangle)^2$$

$$= \langle \psi | (\omega - \langle \omega \rangle)^2 | \psi \rangle$$

Compatible and Incompatible Variables

In a given state $\mid \psi \rangle$ one cannot say unambiguously a particle has a definite value for a dynamical variable $\omega$. A measurement can yield any eigenvalue for which $\langle \psi | \omega \rangle$ is non-zero. A particle in one state $\mid \psi \rangle$. A particle in one of these states has a value $\omega$, $\Delta \omega$ since a measurement yields a fixed value with certainty. We wish to extend these ideas to more than one variable. Consider $n$ variables $\omega_i$ known to move straightforward.
Questions

1) Can we take an ensemble of particles in some state $|\psi\rangle$ and produce a state well defined values of $S_z$ and $\lambda$ for two variables $S, \lambda$?

2) If we do the procedure to produce a state with defined value $w$ and if variables $S, \lambda$ what is probability that we end up with the particular eigenvalue if we start with some state $|\psi\rangle$.

Try first measure $S_z$ on ensemble denoted by $|\psi\rangle$ and take the particles that yield result $w$. Now we have an ensemble of particles in state $|\lambda\rangle$ and in general they are not eigenvalues of $A$ with eigenvalue $\lambda$. Again performing a second test that is denoted as measurement of $S_z$ in general the state is disturbed unless $|\lambda\rangle$ is also an eigenvalue of $A$ with eigenvalue $\lambda$. Let $|\psi\rangle$ be such state.

$$S_z |\psi\rangle = |\lambda\rangle$$

$$\lambda |\psi\rangle = \lambda |\lambda\rangle$$

So \( (S_z - \lambda \mathbb{1}) |\psi\rangle = 0 \)

Thus \( (S_z - \lambda \mathbb{1}) \) must have zero eigenvalues if zero eigenvalues of simultaneous eigenvalues are to exist. Apart of operator $S_z$ will fall into 3 classes.

A) Compatible: \( [S_z, A] = 0 \)
B) Incompatible: \( [S_z, A] = \text{something with no zero eigenvalue} \)
C) Not A or B.
Case A: A complete family of simultaneous eigenevectors exist. Each element \( |\psi\rangle \) of the linear vector space determined by \( (\theta, \psi, \Delta) \) is a multiple of \( \Delta = 1 \).

Case B: The most famous example of this is provided by position and momentum. Let \( p = i/\hbar \). So the commutation relation \( [X, P] = i\hbar \) is non-null state. There is no way to have a simultaneous eigenvector of \( X \) and \( P \). Beyond Heisenberg uncertainty principle.

Case C: Some states are simultaneous eigenvectors.

Now let's talk about second quantization protocols.

**Case A**: Begin by assuming no degeneracy. Then \( \rho(\lambda) = |\langle w\lambda | \phi \rangle|^2 \).

Now let's consider a subbasis of \( \mathbb{V} \) spanned by \( |w_1\rangle, |w_2\rangle, |w_3\rangle \). Consider a non-degenerate state in this subbasis.

\[
|\psi\rangle = \alpha |w_1\rangle + \beta |w_2\rangle + \gamma |w_3\rangle
\]

Suppose we measure \( \Omega \) and find state \( w_3 \). The state becomes \( |w_3\rangle \). Then \( \rho(w_3, \lambda) = \lambda \).

Then \( \rho(w_2, \lambda) = \rho(w_3, w_2) \).

Suppose the measurement of \( \Omega \) gave \( w_1 \). The state becomes \( |w_1, \lambda\rangle \). Now suppose we do measurement in order. First measure \( \lambda \) to get \( \lambda \). Then state after measurement:

\[
|\psi\rangle = \frac{\beta |w_1\rangle + \gamma |w_2\rangle}{\sqrt{\beta^2 + \gamma^2}}
\]
Polarizable of chi qute. \( P(\chi) = \beta^2 + \chi^2 \), with
sum of polarizables of qute with \( \chi \).

\[
\left( \begin{array}{c} Y_1 \cr Y_2 \end{array} \right) = \left( \begin{array}{c} \beta^2 + \chi^2 \cr \beta^2 + \chi^2 \end{array} \right)
\]

Next we measure \( S_2 \) at a value of \( \chi \).

\[
P(\chi) = \frac{\beta^2 + \chi^2}{\beta^2 + \chi^2} = \frac{1}{\beta^2 + \chi^2}
\]

So \( P(2, \chi) = P(\chi, 2) \) independent of degeneracy. Note that the actual measurement did disturb the slab in this case.

So now you know how to prepare a defect slab.

End measure \( S \) at a value \( \chi \).

If the spacey side degenerates, it is non-degenerate for
there means \( 1' \rightarrow 1' \).

If not, degenerate.

If other side measure another possible

\( \chi \) and obtain \( 10', 10' \) if degenerate.

Subtype of regular if \( \Sigma \).

If still degenerate

add a new measure \( \Sigma + \Delta \) obtain \( 10', 10' \).

If non-degenerate, done. If not keep going. Eventually reach a unique slab.

That is true for a complete set of connecting observables under a
unique common assumption.

Note even if \( S_2, A \) are measurable can
compute the probability of find another segment
of \( \Sigma + \Delta \) or another \( S_2 \).

In general, also does not reduce a probability
of well defined \( S_2, A \).