Phys 125a: Homework 4. Due Nov. 2, 2005

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Problem 1

Exercise 7.4.2. Find $\langle X \rangle$, $\langle P \rangle$, $\langle X^2 \rangle$, $\langle P^2 \rangle$, $\Delta X \cdot \Delta P$ in the harmonic oscillator energy eigenstate $|n\rangle$.

Problem 2

Exercise 7.4.5. At $t = 0$ a particle starts out in $|\psi(0)\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$, where $|0\rangle$ and $|1\rangle$ are harmonic oscillator energy eigenstates.

a. Find $|\psi(t)\rangle$.

b. Find $\langle X(0) \rangle = \langle \psi(0)|X|\psi(0)\rangle$, $\langle P(0) \rangle$, $\langle X(t) \rangle$, $\langle P(t) \rangle$.

c. Find $\langle \dot{X}(t) \rangle$ and $\langle \dot{P}(t) \rangle$ using Ehrenfest’s theorem and solve for $\langle X(t) \rangle$ and $\langle P(t) \rangle$. Compare with the results of part b.

Problem 3

Exercise 7.4.9. Consider the unconventional (but fully acceptable) operator choice

$$
X \rightarrow x \\
P \rightarrow -i\hbar \frac{d}{dx} + f(x)
$$

in the $X$ basis.

a. Verify that the canonical commutation relation is satisfied.

b. It is possible to interpret the change in the operator assignment as a result of a unitary change of the $X$ basis:

$$
|x\rangle \rightarrow |\tilde{x}\rangle = e^{ig(x)/\hbar} |x\rangle = e^{ig(x)/\hbar} |x\rangle,
$$

where

$$
g(x) = \int^x f(x')dx'.
$$

First verify that

$$
\langle \tilde{x}|X|\tilde{x}'\rangle = x\delta(x - x'),
$$

i.e.,

$$
X \rightarrow x
$$

in the new $X$ basis.

Next verify that

$$
\langle \tilde{x}|P|\tilde{x}'\rangle = \left[ -i\hbar \frac{d}{dx} + f(x) \right] \delta(x - x')
$$
This exercise teaches us that the “X basis” is not unique; given a basis $|x\rangle$, we can get another $|\tilde{x}\rangle$, by multiplying by a phase factor which changes neither the norm nor the orthogonality. The matrix elements of $P$ change with $f$, the standard choice corresponding to $f = 0$. Since the presence of $f$ is related to a change of basis, the invariance of the physics under a change in $f$ (from zero to nonzero) follows. What is novel here is that we are changing from one $X$ basis to another $X$ basis rather than to some other $\Omega$ basis. Another lesson to remember is that two different differential operators $\omega(x, -i\hbar d/dx)$ and $\omega(x, -i\hbar d/dx + f)$ can have the same eigenvalues and a one-to-one correspondence between their eigenfunctions, since they both represent the same abstract operator $\Omega(X, P)$.

\textit{Problem 4}

The Nobel Prize in Physics for 2005 has been awarded in part to Roy Glauber for his work on the properties of the so-called “coherent states” of the harmonic oscillator, particularly as they apply to the quantum theory of optics.\footnote{See \url{http://nobelprize.org/physics/laureates/2005/index.html}.} In this problem you will work a few of the basic properties of the coherent state (forty years too late, alas, for it to be Nobel Prize-winning material).

A coherent state $|\lambda\rangle$ of a one-dimensional simple harmonic oscillator is defined to be an eigenstate of the destruction operator $\hat{a}$:

$$\hat{a}|\lambda\rangle = \lambda|\lambda\rangle .$$

(Notice that, since $\hat{a}$ is not Hermitian, $\lambda$ will in general be a complex number, and the coherent states will \textit{not} form an orthonormal basis. Notice also that the ground state $|0\rangle$ is a coherent state with $\lambda = 0$.)

\textbf{a.} Prove that

$$|\lambda\rangle = e^{-|\lambda|^2/2} \cdot e^{\lambda \hat{a}^\dagger}|0\rangle$$

is a normalized coherent state.

\textit{Hint:} You may try first writing the second exponential as a sum of powers
of $\hat{a}^\dagger$, and then expressing $|\lambda\rangle$ as a sum over the energy eigenstates $|n\rangle$.

b. Let:

$$f(n) \equiv \langle n|\lambda \rangle.$$  

Show that $|f(n)|^2$ has the form of a Poisson distribution:

$$|f(n)|^2 = \frac{\nu^n}{n!} e^{-\nu},$$

with $\nu$ a real, positive parameter. Give $\nu$ in terms of $\lambda$.

c. Prove the minimum uncertainty relation for such a state (i.e., show that $|\lambda\rangle$ saturates the lower bound of the Heisenberg uncertainty relation).

d. Using the Hamiltonian for the simple harmonic oscillator, $\hat{H} = \hbar \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2})$, show that after a time $t$ the state $|\lambda\rangle$ will evolve into another coherent state $|\lambda'\rangle$. Give $\lambda'$ in terms of $\lambda$, $\omega$, and $t$, and indicate the overall phase of the new coherent state.

\[\text{--- Footnote ---}\]

This result is key to Glauber’s work. In quantum electrodynamics, each frequency mode of the electromagnetic field is a simple harmonic oscillator. If a mode is described by a coherent state, $|f(n)|^2$ gives the probability of counting $n$ photons of that frequency. The Poisson distribution for $|f(n)|^2$ matches the classical result for the number of pulses produced by a photodetector. This is part of the reason why coherent states are the closest quantum analog to classical optics, and why they are therefore so useful in understanding the classical limit of quantum optics.