Phys 125a: Homework 5. Due Nov. 23, 2005

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Problem 1 (Exercises 10.3.1, 10.3.2, and 10.3.3)

a. Two identical bosons are found to be in states $|\phi\rangle$ and $|\psi\rangle$. Write down the normalized state vector describing the system when $\langle\phi|\psi\rangle \neq 0$.

b. When an energy measurement is made on a system of three bosons in a box, the \( n \) values obtained were 3, 3, and 4. Write down a symmetrized, normalized state vector.

c. Imagine a situation in which there are three particles and only three states \( a, b, \) and \( c \) available to them. Show that the total number of allowed, distinct configurations for this system is

(1) 27 if they are distinct
(2) 10 if they are bosons
(3) 1 if they are fermions

Problem 2 (Exercises 10.2.1 and 10.2.2)

a. Recall that a particle in a one-dimensional box extending from \( x = 0 \) to \( L \) is confined to the region \( 0 \leq x \leq L \); its wave function vanishes at the edges \( x = 0 \) and \( L \) and beyond. Consider now a particle confined in a three-dimensional cubic box of volume \( L^3 \). Choosing as the origin one of its corners, and the \( x, y, \) and \( z \) axes along the three edges meeting there, show that the normalized energy eigenfunctions are

$$ \Psi_E(x, y, z) = \left( \frac{2}{L} \right)^{1/2} \sin\left( \frac{n_x \pi x}{L} \right) \left( \frac{2}{L} \right)^{1/2} \sin\left( \frac{n_y \pi y}{L} \right) \left( \frac{2}{L} \right)^{1/2} \sin\left( \frac{n_z \pi z}{L} \right) $$

where
$$ E = \frac{\hbar^2 \pi^2}{2ML^2} (n_x^2 + n_y^2 + n_z^2) $$

and \( n_i \) are positive integers.

b. Quantize the two-dimensional oscillator for which the classical Hamiltonian is,

$$ H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m \omega_x^2 x^2 + \frac{1}{2} m \omega_y^2 y^2 $$

Show that the allowed energies are

$$ E = (n_x + 1/2) \hbar \omega_x + (n_y + 1/2) \hbar \omega_y, \quad n_x, n_y = 0, 1, 2, \ldots $$

Write down the corresponding wave functions in terms of single oscillator wave functions. Verify that they have definite parity (even/odd) number \( x \to -x, y \to -y \) and that the parity depends only on \( n = n_x + n_y \).

c. Consider next the isotropic oscillator \( (\omega_x = \omega_y) \). Write explicit, normalized eigenfunctions of the first three states (that is, for the cases \( n = 0 \) and \( 1 \)). Reexpress your results in terms of polar coordinates \( \rho \) and \( \phi \). Show that the degeneracy of a level with \( E = (n + 1) \hbar \omega \) is \( n + 1 \).

Problem 3

Let \( \hat{A} \) and \( \hat{B} \) be arbitrary operators. Define a new operator \( \hat{f} \) as:

$$ \hat{f}(x) \equiv \exp\left( x\hat{A} \right) \hat{B} \exp\left( -x\hat{A} \right), $$
where $x$ is a $c$–number variable. Let primes denote derivation with respect to $x$.

a. Show that:
\[ \hat{f}'(x) = \exp \left( x\hat{A} \right) [\hat{A}, \hat{B}] \exp \left( -x\hat{A} \right) \]
and find $\hat{f}''(x)$.

b. Expanding $\hat{f}(x)$ as a Taylor series, argue that:
\[ \hat{f}(x) = \hat{B} + x [\hat{A}, \hat{B}] + \frac{x^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \ldots \]

c. Use the result of part b to show that:
\[ e^{x\hat{B}/\hbar} \hat{X} e^{-x\hat{B}/\hbar} = \hat{X} + \ell. \]

d. Now define the operator:
\[ \hat{C}(x) \equiv e^{x\hat{A}} e^{x\hat{B}}. \]
Show that $\hat{C}'(x) = \hat{O}(x)\hat{C}(x)$ for some operator $\hat{O}(x)$. Use the result of part b to give $\hat{O}(x)$ as a power series in $x$.

e. Now suppose that $[\hat{A}, \hat{B}]$ commutes with $\hat{A}$, so that all but the first two terms in the expansion for $\hat{O}(x)$ disappear. Argue that this implies that:
\[ \hat{C}(x) = \exp \left\{ x \left( \hat{A} + \hat{B} \right) + \frac{x^2}{2} [\hat{A}, \hat{B}] \right\}. \]
f. Using the result for part e, show the so-called **Campbell-Baker-Hausdorff theorem**: if $[\hat{A}, \hat{B}]$ commutes with both $\hat{A}$ and $\hat{B}$, then:
\[ e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A}, \hat{B}]/2}. \]

**Problem 4**

Recall your work on coherent states from Homework 4: a coherent state $|\lambda\rangle$ of the simple harmonic oscillator is an eigenstate of the destruction operator $\hat{a}$, with complex eigenvalue $\lambda$.

a. Show that a space translation by a finite distance $x_0$ takes the vacuum state $|0\rangle$ of a simple harmonic oscillator to a coherent state. Give the eigenvalue $\lambda$ of the coherent state in terms of $x_0$.

This process only allows you to generate coherent states with purely real eigenvalues. The general coherent state can be obtained by displacing the vacuum $|0\rangle$ along both the $X$ and the $P$ directions in phase-space. To do this, we define the displacement operator:
\[ D(x_0, p_0) \equiv \exp \left\{ \frac{i}{\hbar} \left( p_0 \hat{X} - x_0 \hat{P} \right) \right\} \]

b. Show that we can equivalently write this displacement operator as:
\[ D(\lambda) = \exp \left[ \text{Re}(\lambda) (\hat{a}^\dagger - \hat{a}) + i \text{Im}(\lambda) (\hat{a} + \hat{a}^\dagger) \right] = e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}, \]
where $x_0 = \sqrt{2\hbar/m \omega} \text{Re}(\lambda)$ and $p_0 = \sqrt{2\hbar/m \omega} \text{Im}(\lambda)$. 
c. Show that the formula for a coherent state

$$|\lambda\rangle = D(\lambda)|0\rangle$$

is equivalent to the form that we gave you at the beginning of problem 4 in Homework 4.

**Hint:** First write $D(\lambda) = e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}$ and then use the Campell-Baker-Hausdorff theorem from Problem 3.