Problem 1

Exercise 14.4.3. A spin $\frac{1}{2}$ particle starts out in the spin state $\left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ in the $S_z$ basis. It’s subject to an external magnetic field

$$\vec{B}(t) = B \cos(\omega t) \hat{x} - B \sin(\omega t) \hat{y} + B_0 \hat{z},$$

where $B \ll B_0$.

The state obeys Schrödinger’s equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi\rangle,$$

with $H = -\gamma \vec{S} \cdot \vec{B}$.

Classical reasoning suggests that, in the frame that rotates along with the $x$-$y$ component of the external $B$–field, the Hamiltonian will be time-independent and that the time-evolution of the spin state should be governed by

$$\vec{B}_r = B \hat{x}_r + \left( B_0 - \frac{\omega}{\gamma} \right) \hat{z},$$

where $\hat{x}_r$ is the unit vector in the $x$–direction in the rotating frame.\(^1\)

Consider, therefore, the state $|\psi_r(t)\rangle$, related to $|\psi(t)\rangle$ by a time-dependent rotation:

$$|\psi_r(t)\rangle = e^{-i\omega r S_z/\hbar} |\psi(t)\rangle.$$ \hspace{1cm} (4)

Combine Eqs. (2) and (4) to derive Schrödinger’s equation for $|\psi_r(t)\rangle$ in the $S_z$ basis and verify that the classical expectation is borne out. Solve for $|\psi_r(t)\rangle = U_r(t) |\psi_r(0)\rangle$ by computing $U_r(t)$, the propagator in the rotating frame. Rotate back to the lab frame and show that this gives, in the $S_z$ basis,

$$|\psi(t)\rangle = \left( \begin{array}{c} \cos \left( \frac{\omega_r t}{2} \right) + i \frac{\omega_0 - \omega}{\omega_r} \sin \left( \frac{\omega_r t}{2} \right) e^{i\omega t/2} \\ i \frac{\omega_B}{\omega_r} \sin \left( \frac{\omega_r t}{2} \right) e^{-i\omega t/2} \end{array} \right),$$

where $\omega_0 \equiv \gamma B_0$ and $\omega_r \equiv \gamma \sqrt{B^2 + \left( B_0 - \frac{\omega}{\gamma} \right)^2}$.

Compare this to the initial spin state and see what is happening to the spin in the case $\omega = \omega_0$.

\(^1\)For this problem it might be useful to read Shankar’s discussion of parametric resonance (p. 392-4), which wasn’t covered in lecture.
Problem 2

a. Exercise 15.2.2. Find the Clebsh-Gordan coefficients of $\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$ and of $1 \otimes 1 = 2 \oplus 1 \oplus 0$.

b. Exercise 15.2.3. Argue that $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$.

Problem 3

Exercise 15.2.5.

a. Show that $P_1 = \frac{3}{4}I + \left( \vec{S}_1 \cdot \vec{S}_2 \right) / \hbar^2$ and $P_0 = \frac{1}{4}I - \left( \vec{S}_1 \cdot \vec{S}_2 \right) / \hbar^2$ are projection operators. That is, show that they obey $P_i P_j = \delta_{ij} P_j$. (Hint: Use the identity $(\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = (\vec{A} \cdot \vec{B}) I + i (\vec{A} \times \vec{B}) \cdot \vec{\sigma}$.)

b. Show that these project into the spin-1 and spin-0 spaces for $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$.

Problem 4

Exercise 15.2.7. Show that when we add $j_1$ to $j_1$, the states with $j = 2j_1$ are symmetric. Show that the states with $j = 2j_1 - 1$ are antisymmetric. (Hint: Argue for the symmetry or anti-symmetry of the top states and show that lowering doesn’t change it.) This pattern of alternating symmetry continues as $j$ decreases, but is harder to prove.