Review of Electrodynamics Classical

Response of matter to electromagnetic fields is given by laws of force law

\[ F = q \left( \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) \]

For particles of charge \( q \), moving with \( \vec{v} \), response of fields to charges is given by Maxwell's equations:

1. \[ \nabla \cdot \vec{E} = \frac{4\pi j}{c} \]
2. \[ \nabla \times \vec{E} + \frac{1}{c^2} \frac{\partial \vec{B}}{\partial t} = 0 \]
3. \[ \nabla \cdot \vec{B} = 0 \]
4. \[ \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{j} \]

where \( \vec{E} \) and \( \vec{j} \) are the electric and current densities. They satisfy the continuity equation:

\[ \frac{\partial \vec{E}}{\partial t} + \nabla \cdot \vec{j} = 0 \]

Potentials \( \vec{A} \) and \( \phi \) are introduced as follows. Since \( \nabla \cdot \vec{B} = 0 \), we can write:

\[ \vec{B} = \nabla \times \vec{A} \]

From vector potential \( \vec{A} \), we have:

\[ \nabla \times (\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t}) = 0 \]

So we can write:

\[ \vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\nabla \phi \]
In some potential $\phi$. Now we can write
\[ \mathbf{E} = \nabla \phi \]
\[ \mathbf{B} = \nabla \times \mathbf{A} \]
\[ \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \]

Using $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ we get
\[ -\nabla^2 \mathbf{A} - \frac{1}{c} \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{4\pi}{c} \mathbf{j} \]
\[ \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{1}{c} \frac{\partial}{\partial t} \left( -\frac{\nabla^2 \phi}{c} - \frac{\partial \mathbf{A}}{\partial t} \right) \]
\[ \nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} = -\frac{4\pi j}{c} \]

Now, there is a certain ambiguity on $\mathbf{A}$ and $\phi$.

Suppose we have
\[ \mathbf{A}' = \mathbf{A} - \nabla \Lambda \]
\[ \phi' = \phi + \frac{1}{c} \frac{\partial \Lambda}{\partial t} \]

Then we have $\mathbf{E}$, $\mathbf{B}$ fields
\[ \mathbf{B}' = \nabla \times \mathbf{A}' = \mathbf{B} \]
\[ \mathbf{E}' = -\nabla \phi' - \frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t} \]
\[ = -\nabla \left( \phi + \frac{1}{c} \frac{\partial \Lambda}{\partial t} \right) - \frac{\nabla^2 \Lambda}{c} \]
\[ + \frac{\partial}{\partial t} \left( \frac{\nabla^2 \Lambda}{c} \right) = \mathbf{E} \]
We can now take \( \phi = 0 \). For example take
\[
\Lambda(t, t') = \phi(t, t') \int \phi(t, t') dt'
\]
and that in some \( t' \), \( \phi = 0 \) we can also choose \( \nabla \cdot A = 0 \) by making a time independent gauge factor

\[
\nabla^2 \Lambda = (\nabla \cdot A)
\]

so

\[
\Lambda = -\frac{1}{4\pi} \int d^3 r' \frac{\Lambda(t, r')}{|r - r'|}
\]

and

\[
\nabla^2 \frac{1}{|r - r'|} = -4\pi \delta(r - r')
\]

In constant gauge equations of motion with no sources

\[
\nabla^2 A - \frac{1}{2} \frac{\partial^2 A}{\partial t^2} = 0
\]

\[
\nabla \cdot A = 0
\]

\[
\nabla \times A = 0
\]

\[
\overline{A} = \bar{A}_0 \cos (k \cdot \overline{r} - wt)
\]

where

\[
\omega^2 = k^2
\]

is a solution to that eqn. \( \nabla A = 0 \)

gauge condition means also

\[
\overline{k} \cdot \bar{A}_0 = 0 \quad \bar{A}_0 \perp \overline{k}
\]

and \( \overline{k} \cdot \bar{A}_0 = \nabla \cdot \bar{A}_0 = 0 \) as well. Now let go back and find \( \bar{E}, \bar{B} \) fields

\[
E = -\frac{1}{2} \frac{\partial \bar{A}}{\partial t} - (\overline{w} \overline{A}) \sin (k \cdot \overline{r} - wt)
\]

\[
\bar{B} = \nabla \times \bar{A} = -k \times \bar{A}_0 \sin (k \cdot \overline{r} - wt)
\]
Note that $|\vec{E}| = |\vec{B}|$. Recall for electromagnetics that

$$|\vec{E}| = \frac{c}{\gamma} |\vec{E}_x\vec{B}|$$

gives energy flow across a unit area $\frac{dE}{dt}$ per unit time onto

$$\mathcal{E} = \frac{1}{8\pi} (|\vec{E}|^2 + |\vec{B}|^2)$$

gives the energy density (energy per unit volume) stored in electromagnetic fields for $k_0$.

$$|\vec{E}| = \frac{w}{8\pi c} |\vec{A}|^2$$

$$\mathcal{E} = \frac{1}{8\pi} \frac{w^2}{c^2} |\vec{A}|^2$$

Gauge invariance and potentials in QM

$$\langle \bar{\psi}(\vec{r}, t) \psi(\vec{r'}, t') \rangle = \delta(\vec{r} - \vec{r'}, t - t')$$

$$= N \sum_{\text{paths}} \exp \left( \frac{iS_{\text{classical}}}{\hbar} \right)$$

(normalization)

Now for a particle interacting with electromagnetic field

$$\mathcal{L} = -\frac{1}{2} m \vec{\dot{\psi}}^2 - \frac{e}{c} \vec{\psi} \vec{A}$$
\[
S = \int_{t_1}^{t_2} dt \left( \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \mathbf{A} \cdot \mathbf{A} - q \phi \right)
\]
evaluated along a path \( \mathbf{r} = (\mathbf{r}(t), \phi(t)) \) and \( (t, \mathbf{x}) \)
Now suppose we perform a gauge transformation on the potential \( \phi \to \phi + \frac{d}{dt} \mathbf{A} \cdot \mathbf{A} \)
\[S \to S' = S - \int_{t_1}^{t_2} dt \left( \mathbf{v} \cdot \mathbf{A} + \frac{d}{dt} \mathbf{A} \right)\]
but \( \mathbf{v} \cdot \mathbf{A} + \frac{d}{dt} \mathbf{A} = \partial \mathbf{A} / \partial t \)
\[S \Rightarrow S' = S \left( \mathbf{A}(t_1) - \mathbf{A}(t_2) \right) \]
S = S' only if the same classical dynamics.

We only differ by a boundary term \( \partial \mathbf{A} / \partial t \).

Equivalent to a change of basis:
\[ |\tilde{\phi}' \rangle = e^{-i q \mathbf{A} \cdot \mathbf{r} / \hbar} |\phi \rangle \]
Cloning an act
\[ \mathbf{H}(\tilde{\phi}') = \langle \tilde{\phi}' | \mathbf{H} | \phi \rangle = e^{-i \mathbf{A} \cdot \mathbf{r} / \hbar} \mathbf{H} e^{i \mathbf{A} \cdot \mathbf{r} / \hbar} \]
Well its just a change in measurement.
No $B$ field

\[ y(t) = y_1(t) + y_2(t) \]

This is without $B$ field. Now we can add an $B$ field. Each path gets an extra

\[ \exp \left[ \frac{ig}{nc} \int_{t, t'} \mathbf{A} \cdot d\mathbf{l} \right] \]

\[ = \exp \left[ \frac{ig}{nc} \int_{\text{source}} \mathbf{A} \cdot d\mathbf{s} \right] \]

Set $\mathbf{V} \times \mathbf{A} = 0$ near $P_1, P_2$. The integral over some path $P_1$ and its nearby path $P_2$ is well defined. But

\[ \int_{P_2} \mathbf{A} \cdot d\mathbf{s} - \int_{P_1} \mathbf{A} \cdot d\mathbf{s} = \oint \mathbf{A} \cdot d\mathbf{s} \]

\[ = \int_{\mathbf{S}} (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint \mathbf{B} \cdot d\mathbf{s} = \mathbf{E} \neq 0 \]
With \( \text{field} \)

\[
y_i(\vec{r}) = \exp\left(\frac{ie}{\hbar c} \int_{\vec{r}_1}^{\vec{r}} A \cdot d\vec{r}'\right) y_i(\vec{r}) \\
+ \exp\left(\frac{ie}{\hbar c} \int_{\vec{r}_2}^{\vec{r}} A \cdot d\vec{r}'\right) y_i(\vec{r})
\]

\[
= \left[ \text{overall factor} \right] \left( y_i(\vec{r}) + \exp\left(\frac{ie}{\hbar c} \int A \cdot d\vec{r}\right) y_i(\vec{r}) \right)
\]

\[
= \left[ \text{overall factor} \right] \left( y_i(\vec{r}) + \exp\left(\frac{ie}{\hbar c} \int A \cdot d\vec{r}\right) y_i(\vec{r}) \right)
\]

\[\frac{\Phi}{\hbar c} = 2\pi n\]

The effect comes from the \( |y_i(\vec{r})|^2 \)

But the dot product

\[\frac{\Phi}{\hbar c} = 2\pi n\]
Magnetic Monopoles

The Maxwell equations hold similarly for $E, B$ and there are no magnetic charges just electric charges. The source of a magnetic field is a moving electric charge (an electromagnetic source). Suppose static magnetic charge source well density $q_m$. Then we would have

$$\nabla \cdot B = 4\pi q_m$$

analogous to

$$\nabla \cdot E = 4\pi q$$

Suppose there is a point magnetic charge at the vertex of length $q_m$, analogous to electric charge

$$B = \left( \frac{q_m}{r^2} \right) \hat{r}$$

Now the volt potential plays an important role particularly in QM. Axial potential might appear, can derive the magnetic field from

$$\mathbf{A} = \frac{q_m (1 - \cos \theta)}{\sin \theta} \hat{\phi}$$

not singular at $\theta = 0$

using spherical polar coordinates. Recall the

in spherical polar coordinate

$$\nabla \times \mathbf{A} = \hat{\phi} \left[ -\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (A_{\phi} \sin \theta) \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( r A_{\phi} \right) \right] + \hat{\theta} \left[ -\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( r A_{\theta} \right) \right] + \hat{r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r A_{\phi} \right) \right]$$
But even though vector potential is not unique at $\Theta = 0$, it is singular at $\Theta = \pi$ on a regular realm. Cannot have a free of singular everywhere! Gauss law yields

$$\int \mathbf{B} \cdot d\mathbf{a} = 4\pi I$$

closed surf

for any surface around origin. On the other hand, $A$ non-singular except $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

$$\int \mathbf{B} \cdot d\mathbf{a} = \int \nabla \cdot (\nabla \times \mathbf{A}) \, d\mathbf{x} = 0$$
closed surf volume

How the such a singular not cause any flux so $\int \mathbf{B} \cdot d\mathbf{a} = 0$

\[ \text{[Diagram: Monopole, String, Caricatured Flux]} \]

Flat

$$I = \int \mathbf{B} \cdot d\mathbf{a} = 4\pi I c$$

Want seem to be understood so I has no physical effect. Born at horizon string flux must causal. Forbidden

$$\frac{4\pi e c m}{\hbar c} = 2\pi I$$

$n = 0, \pm 1, \pm 2, \ldots$

$m = \frac{(n + I c)}{\hbar c}$

magnetic charge

related to electric

Dirac Quantization Conjecture