Problem 4 requires a large number of commutators from Zwiebach Chapter 12, as well as a few that are readily derived from these. There is also a trick, without which the problem is significantly more difficult. Below, you will find a collection of all of the useful commutators, as well as a description of the trick.

1. Useful commutators

The following commutators appear in Chapter 12 of Zwiebach:

\[
[x^I_0, p^J] = i\delta^{IJ}, \quad [x^-_0, p^+] = -i
\]  
\[
[\alpha^I_m, \alpha^J_n] = m\delta^{IJ}\delta_{m+n,0}
\]  
\[
[L^+_m, \alpha^I_n] = -n\alpha^J_{m+n}
\]  
\[
[L^+_m, x^I_0] = -i\sqrt{2}\alpha'\alpha^I_m
\]  
\[
[L^+_m, L^+_n] = (m-n)L^+_{m+n} + \frac{D-2}{12}(m^3 - m)\delta_{m+n,0}.
\]

Since \(\alpha^J_0 = \sqrt{2}\alpha'p^J\), Eqs. (2) and (3) imply that

\[
[\alpha^I_m, p^J] = 0, \quad [L^+_m, p^J] = 0.
\]

Similarly, note the following vanishing commutators:

\[
[x^I_0, \alpha^J_m] = 0 \quad \text{for} \quad m \neq 0 \quad \text{(cf. Eq. (1) for} \ m = 0),
\]  
\[
[O, p^+] = 0 \quad \text{for} \quad O = x^I_0, p^I, \alpha^I_m, L^+_m,
\]  
\[
[O, x^-_0] = 0 \quad \text{for} \quad O = x^I_0, p^I, \alpha^I_m, L^+_m.
\]

Using Eqs. (3) and (5), one can also show that for arbitrary \(m\),

\[
[L^+_0, L^+_{-m}\alpha^J_m] = 0,
\]  
\[
[L^+_0, \alpha^J_{-m}L^+_m] = 0,
\]  
\[
[L^+_0, \alpha^I_{-m}\alpha^I_m] = 0.
\]
In fact, the last three commutators are special cases of the following more general result that follows from $[L_0^\perp, \alpha_n] = -n\alpha_n$ and $[L_0^\perp, L_n^\perp] = -n L_n^\perp$:

$$[L_0^\perp, (\text{any string of } L_m^\perp \text{ and } \alpha_n^\perp)] = -(\text{sum of subscripts})(\text{same string}).$$  

(13)

However, this is not needed for Problem 4.

Finally, the the rules for evaluating commutators of products are

$$[A, BC] = [A, B]C + B[A, C], \quad (14)$$


2. The trick.

Now here is the trick. Your result will contain terms in the $\alpha_m^\perp$ both of quadratic order $\alpha\alpha$ and quartic order $\alpha\alpha\alpha\alpha$, treating the Virasoro operators $L^\perp = \sum \alpha\alpha$ as quadratic order. As long as you normal-order the quartic terms, they automatically sum to zero by the classical result! The nonzero quantum commutators just alter the classical result by terms quadratic in the $\alpha_m$.

This saves an enormous amount of work—it is extremely difficult to check that the quartic terms vanish by explicit computation, so this is not recommended. Instead you can adopt the following procedure.

**Step 1.** Take your result with both quadratic and quartic terms, and then use the commutation relations to express the result in normal-ordered form, that is, with all of creation operators $\alpha_{-m}$ for $m > 0$ on the left in each term, all the annihilation operators $\alpha_m$ for $m > 0$ on the right, and a possible Virasoro operator $L_n^\perp$ in the middle. The position of $\alpha_0$ is unimportant since $\alpha_0$ commutes with $L_n^\perp$ and $\alpha_n$. (In your result, the quartic terms of the form $L_m^\perp L_n^\perp$ with two Virasoro operators should cancel, so there is no need to consider the normal ordering of such terms). In other words, express your result as a sum of terms like the following, which are normal-ordered by the rules just described:

$$\alpha_{-m}\alpha_n, \quad \alpha_{-m}\alpha_{-n}L_p^\perp, \quad \alpha_{-m}L_p\alpha_n, \quad \text{and } L_p^\perp\alpha_m\alpha_n \quad \text{for } m, n \geq 0. \quad (16)$$

**Step 2.** Simply drop all the quartic terms, since we know they sum to zero by the classical result. That is, retain terms like the first term in Eq. (16) and drop terms like the other three.

What is left should be Zwiebach in Eq. (12.162), corrected by an overall minus sign on the right hand side.